A modular ordinal analysis of metapredicative subsystems of second order arithmetic

Habilitationsschrift

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Contents

Introduction

This thesis leaps into the metapredicative and puts forward a type of ordinal analysis that goes without "impredicative methods". Neither does our approach involve the use of collapsing functions nor uncountable ordinals. Furthermore, the rules of the infinitary systems used to eliminate cuts are intrinsically sound, as opposed to e.g. infinitary systems equipped with an Ω-rule. We are building – bottom-up – stronger and stronger theories by applying successively stronger operations which we have already understood to theories which we have already understood.

Metapredicative ordinal analysis

Metapredicative theories is a term coined by Jäger and his research group in Bern. A first description of this notion is found in Strahm's paper "First steps into metapredicativity in explicit mathematics" [27]:

> Metapredicativity is a new general term in proof theory which describes the analysis and study of formal systems whose proof-theoretic strength is beyond the Feferman-Schütte ordinal Γ_0 but which are nevertheless amenable to purely predicative methods.

The distinguishing feature of a *metapredicative theory* is that its ordinal analysis can be performed by abstaining from so-called *impredicative* methods. Or as put by Jäger in [6]:

> The collection of metapredicative systems comprises all those theories which are not predicatively reducible and whose proof-theoretic analysis can be carried through without making use of any impredicative methods. [...] Our experience shows that typical impredicative methods always refer to some sort of collapsing techniques and collapsing functions, either directly applied to infinitary proofs or to the ordinals assigned to proofs or to both.

At present, the predicative theories are thoroughly understood, and so is a large segment of impredicative theories with the strength $|\mathsf{ID}_1|$ and beyond. This leaves a huge gap between the Feferman-Schütte ordinal Γ_0 and the Bachmann-Howard ordinal $|\mathsf{ID}_1| = \vartheta\varepsilon_{\Omega+1}$, of which only a small initial segment is charted, namely a couple of theories with a proof-theoretic ordinal up to $\varphi 1000$.

Initial steps into the metapredicative were made by Strahm [27] in the framework of explicit mathematics by analyzing theories of strength $\varphi 1\omega 0$ and $\varphi 1\varepsilon_0 0$. Subsystems of second order arithmetic of corresponding strength, $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$ and $\mathsf{ATR} + (\Sigma^1_1\text{-}\mathsf{DC})$, have been considered by Jäger and Strahm [8]. Independently, Rathjen has analyzed a Martin-Löf type theory with ordinal strength $\varphi 1\Gamma_0 0$ (Rathjen [17, 16]). The metapredicative variants of transfinitely iterated inductive definitions $\widehat{\mathsf{ID}}_\alpha$ were analyzed by Jäger, Kahle, Setzer and Strahm in [7]. Their autonomous closure corresponds to a subsystem of second order arithmetic called FTR_0 (fixed point transfinite recursion) which has ordinal strength $\varphi 200$ (see Strahm [28]).

In a next step, various systems of strength meta-predicative Mahlo – captured by the ordinal $\varphi\omega 00$ – were studied. All these systems are characterized by Π^1_2-reflection on ω-models of ACA_0, Π_2-reflection on admissible sets or some corresponding form of reflection (cf. Jäger and Strahm [9], Strahm [29] and Rüede [19]). Then, Thiel and Gibbons have researched in their dissertations [33, 5] theories with proof-theoretic ordinal $\varphi 1000$ (the Ackermann ordinal). These theories correspond to systems of second order arithmetic where an ω-tower of Mahlo-universes is asserted to exist, i.e. $\mathsf{ACA}_0 + \forall Z \exists X [Z \dot{\in} X \wedge \Pi^1_2\text{-}\mathsf{Refl}(\Sigma^1_1\text{-}\mathsf{DC}_0){\restriction}X]$. And quite recently, in Ranzi [13] and Ranzi and Strahm [14] systems are analyzed whose proof-theoretic ordinal is the small Veblen number $\vartheta\Omega^\omega$ which correspond to the theory $\mathsf{p}_3(\mathsf{ACA}_0)$ (Π^1_3-reflection on ω-models of ACA_0). Besides, some ideas how to tackle the systems $\mathsf{p}_{n+3}(\mathsf{ACA}_0)$ (Π^1_{n+3}-reflection on ω-models of ACA_0) are sketch in Jäger and Strahm [10].

To narrow down the concept of metapredicative ordinal analysis we review the current practice as applied in the aforementioned papers. First, we look at the upper bound computations, then at the well-ordering proofs. Since we are interested in developing a modular ordinal analysis for subsystems of second order arithmetic, we specialize to subsystems of second order arithmetic, although mutatis mutandis our observations apply to metapredicative systems in general.

To describe the general procedure to obtain upper bounds, we denote by T^ϵ the L_2-theory that comprises the logical axioms for classical two-sorted predicate calculus, and axioms for the primitive recursive functions and relations. $\overset{*}{\mathsf{T}}{}^\epsilon$ denotes the corresponding infinitary Tait-style system that derives formulas without free number variables: the ω-rule replaces the $\forall x$-rule, and its axioms are all sequents of the form Γ, A and $\Gamma, B, \neg C$, where A is a true literal, and A and B are numerically equivalent literals (so $\overset{*}{\mathsf{T}}{}^\epsilon$ is a second order version of PA^*). For the time being, T stands for some finitary (formal) theory, and $\overset{*}{\mathsf{T}}$ some infinitary Tait-style system. If the depth of a derivation is less than γ and the cut-rule is restricted[1] to (main formulas of) axioms of $\overset{*}{\mathsf{T}}$ that are not also axioms of T^ϵ, this is indicated by $\overset{*}{\mathsf{T}} \,\frac{<\gamma}{*}\, \Gamma$. An upper bound of T is then a limit ordinal γ, so that each derivation $\mathsf{T} \vdash \Gamma$ of an arithmetical sequent can be transformed into a derivation $\overset{*}{\mathsf{T}}{}^\epsilon \,\frac{<\gamma}{*}\, \Gamma$, which by the above definition is cut-free.

[1]The cut-rule is only applicable if one of the cut-formulas is main formula of an axiom of T.

An upper bound of a theory T is then computed as follows. First, a procedure is given that transforms each derivation $\mathsf{T} \vdash \Gamma$ into an infinitary derivation $\overset{*}{\mathsf{T}} \vdash^{<\gamma} \Gamma$ for some limit ordinal γ. Then, the costs of eliminating cuts in $\overset{*}{\mathsf{T}}$ are figured out using the following means:

(i) Partial cut-elimination: $\overset{*}{\mathsf{T}} \vdash^{\alpha} \Gamma \Longrightarrow \overset{*}{\mathsf{T}} \vdash^{<\varphi 1 \alpha}_{*} \Gamma$.

(ii) Using (i) and other standard techniques from predicative proof-theory such as asymmetric interpretations, $\overset{*}{\mathsf{T}}$ is reduced to the union of suitable[2] intermediate systems $(\overset{*}{\mathsf{T}}_{\xi} : \xi \in I)$ for which the costs of cut-elimination are already known. Thus, for each $\xi \in I$, there is a function f_{ξ}, so that for all $\eta < \xi$ and each sequent $\Gamma \subseteq \mathsf{Fml}(\breve{\mathsf{T}}_{\eta})$, consisting of formulas for which the transformation works,

$$(*) \qquad \overset{*}{\mathsf{T}}_{\xi} \vdash^{\beta}_{*} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}_{\eta} \vdash^{f_{\xi}(\beta)}_{*} \Gamma.$$

Under the additional assumption that $\overset{*}{\mathsf{T}}_0$ is $\overset{*}{\mathsf{T}}^{\epsilon}$ and $\mathsf{Fml}(\overset{*}{\mathsf{T}}^{\epsilon})$ are the arithmetical formulas, and further, there is a function f so that for all $\xi \in I$, if $\beta < \alpha$, then $f_{\xi}(f(\beta)+\omega) < f(\alpha)$, one obtains by induction on α, that for each η (including $\eta = 0$),

$$(\infty\text{-}\mathsf{elim}) \qquad \text{If } \Gamma \subseteq \mathsf{Fml}(\overset{*}{\mathsf{T}}_{\eta}) \text{ and } \bigcup_{\xi \in I} \overset{*}{\mathsf{T}}_{\xi} \vdash^{\alpha}_{*} \Gamma, \text{ then } \overset{*}{\mathsf{T}}_{\eta} \vdash^{f(\alpha)}_{*} \Gamma.$$

Namely, if $\bigcup_{\xi} \overset{*}{\mathsf{T}}_{\xi} \vdash^{\alpha}_{*} \Gamma$ is obtained from $\Gamma, \forall X \neg A(X)$ by a cut with an axiom $\exists X A(X)$ of $\overset{*}{\mathsf{T}}_{\xi}$ for some $\xi > \eta$, then $\forall$-inversion and the I.H. yield $\overset{*}{\mathsf{T}}_{\xi} \vdash^{f(\beta)}_{*} \Gamma, \neg A(U)$ for some $\beta < \alpha$, thus $\overset{*}{\mathsf{T}}_{\xi} \vdash^{f(\beta)+2}_{*} \Gamma$. Now $(*)$ and the assumption on f yield $\overset{*}{\mathsf{T}}_{\eta} \vdash^{f(\alpha)}_{*} \Gamma$. For an arithmetical Γ with $\mathsf{T} \vdash \Gamma$, $\overset{*}{\mathsf{T}}^{\epsilon} \vdash^{<f(\gamma)}_{*} \Gamma$ follows.

With regard to the well-ordering proofs, we see no clear pattern emerging. But all well-ordering proofs seem to use a generalization of Feferman's Lemma 5.3.1. in "Reflection on incompleteness" [3], which states that given a suitable hierarchy $((H^X)_{\beta} : \beta < \gamma)$ of sets above some set X, and given that γ is a limit ordinal which is "sufficiently refined w.r.t. $\alpha < \gamma$", expressed below by $\omega^{\alpha+1}|\gamma$, stating that $\exists \xi (\gamma = \omega^{\alpha+1}\xi)$, then the set

$$\mathcal{C} := \{\alpha : \forall \gamma, \delta \, (\omega^{\alpha+1}|\gamma \wedge \mathsf{Wo}_{\lhd}^{(H^X)_{\lhd\gamma}}(\delta) \rightarrow \mathsf{Wo}_{\lhd}^{(H^X)_{\lhd\gamma}}(\varphi\alpha\delta))\}$$

is progressive w.r.t. $\lhd$. If for each $\alpha < \gamma$ and each set X, there is a hierarchy H^X along $\lhd\restriction\gamma$, then $\mathsf{Wo}_{\lhd}(\gamma)$ together with the progressivity of $\mathcal{C}$ entails $\mathsf{Wo}_{\lhd}(\varphi\gamma 0)$.

[2] If $\eta \leq \xi$, then $\mathsf{Fml}(\breve{\mathsf{T}}_{\eta}) \subseteq \mathsf{Fml}(\overset{*}{\mathsf{T}}_{\xi})$, and each axiom of $\overset{*}{\mathsf{T}}_{\eta}$ is an axiom of $\overset{*}{\mathsf{T}}_{\xi}$; if $\forall X A(X)$ is main formula of an axiom of T_{η}, then $\neg A(U) \in \mathsf{Fml}(\breve{\mathsf{T}}_{\eta})$.

Ad hoc adjustments to the actual demands of the hierarchy, the condition $\omega^{\alpha+1}|\gamma$ and the function $\varphi\alpha\beta$ are used to do the well-ordering proofs.

The analysis of metapredicative systems came to a momentary halt at theories of the strength of the small Veblen number. From our point of view this is because these theories are substantially harder to analyze, and since the currently used methods used in metapredicative ordinal analysis are still unsatisfying in the following respect:

> Carrying out the ordinal analysis of a theory T is a long-winded and strenuous task. All the same, one is content to present as the main result of such labor the proof-theoretic ordinal $|\mathsf{T}|$, although $|\mathsf{T}|$ falls short of subsuming the information gained by its computation. Hence, the invested work cannot be efficiently reused when attempting to analyze stronger systems.

Our modular ordinal analysis presents a solution to this problem. The information gained by carrying out the ordinal analysis of a theory is now efficiently reused, which allows us to compute also the proof-theoretic ordinals of most relevant systems of ordinal strength between the small Veblen number and the Bachmann-Howard ordinal.

Modular ordinal analysis

Although the proof-theoretic ordinal $|\mathsf{T}|$ is of great relevance – it is the extra ingredient needed to proof the consistency of T in $\mathsf{PRA}+\mathsf{QF\text{-}TI}_{\lhd}(|\mathsf{T}|)$ (cf. [2, 4, 11, 15, 21, 22, 31, 32]) – we think that the focus should be shifted towards an understanding of the ordinal analysis of the theory T itself, which in particularly allows to efficiently reuse the information gained in the process.

In order to address the above point, we set-up our modular ordinal analysis as follows.

(i) If $\breve{\mathsf{T}}$ is an L_2-sentence, then T denotes the theory $\mathsf{T} := \mathsf{T}^{\epsilon} + \breve{\mathsf{T}}$ which extends T^{ϵ} by the single axiom $\breve{\mathsf{T}}$. We present all theories under consideration in the form $\mathsf{T} := \mathsf{T}^{\epsilon} + \breve{\mathsf{T}}$. These theories are then manipulated by selected operations Op, functions on sentences, that map a theory T to $\mathsf{Op}(\mathsf{T}) := \mathsf{T}^{\epsilon} + \mathsf{Op}(\breve{\mathsf{T}})$. The focus is on operations that are build from the basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$, where $\mathsf{p}_1(\breve{\mathsf{T}})$ states that above each set Z there is an ω-model X of T, and more generally, $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ expresses Π^1_{n+1}-reflection on ω-models of T.

(ii) The main ideas of modular ordinal analysis are that we can adequately describe a theory T by a sharp bound, a normal function $f_{\mathsf{T}} : \Omega \to \Omega$, and accordingly, that each selected operation Op can be adequately described by a corresponding functional H, to the extend, that if f_{T} is a sharp bound of T, then $H(f_{\mathsf{T}})$ is a sharp bound of $\mathsf{Op}(\mathsf{T})$.

The sharp bound of T, a normal function $f_{\mathsf{T}} : \Omega \to \Omega$, describes T by quantifying the costs of cut-elimination in the corresponding infinitary system $\overset{*}{\mathsf{T}}$, that extends $\overset{*}{\mathsf{T}}{}^{\epsilon}$ axioms Γ, A, where $A \in \mathsf{inst}(\breve{\mathsf{T}})$ is an instance[3] of $\breve{\mathsf{T}}$. It assigns to each limit ordinal γ the least ordinal $f_{\mathsf{T}}(\gamma)$, so that for each finite set Γ of arithmetical L_2^*-formulas (formula without free number variables),

$$\overset{*}{\mathsf{T}} \vdash^{\gamma}_{+} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{f_{\mathsf{T}}(\gamma)}_{*} \Gamma.$$

where, $\overset{*}{\mathsf{T}} \vdash^{\alpha}_{+} \Gamma$ is $\overset{*}{\mathsf{T}} \vdash^{\alpha}_{*} \Gamma$, but allows cuts with additional formulas that do not impede the cut-elimination process and can be eliminated cheaply at a later stage.

A sharp bound f_{T} of T comprises much more information than the proof-theoretic ordinal $|\mathsf{T}|$, which – for the theories we consider in this thesis – can be characterized, for instance, in one of the following ways: $|\mathsf{T}|$ is

(i) the least limit ordinal γ that is not provable in T, i.e. $\mathsf{T} \nvdash \mathsf{Wo}_{\lhd}(\gamma)$,

(ii) the least limit ordinal γ so that for each finite set of arithmetical L_2^*-formulas

$$\mathsf{T} \vdash \Gamma \Longrightarrow \overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{<\gamma}_{*} \Gamma.$$

Since by design of the corresponding infinitary system $\overset{*}{\mathsf{T}}$, $\mathsf{T} \vdash \Gamma$ readily implies $\overset{*}{\mathsf{T}} \vdash^{<\omega}_{*} \Gamma$, (ii) yields that $f_{\mathsf{T}}(\omega) = |\mathsf{T}|$. Clearly, the single value at ω does not yet determine the sharp bound f_{T}. We will see that, for instance, the theories $\mathsf{p}_2(\mathsf{ACA}_0)$ (Σ^1_1-DC_0) and $\mathsf{p}_1^{\omega}(\mathsf{ACA}_0)$ (ACA_0 plus the assertion that for each n, there is an n-tower of ω-models of ACA_0 above any set Z) have both proof-theoretical ordinal $\varphi\omega 0$. Nevertheless, $\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0)$ (ATR_0) has ordinal Γ_0, while $\mathsf{p}_1\mathsf{p}_1^{\omega}(\mathsf{ACA}_0)$ ($\mathsf{p}_1^{\omega+1}(\mathsf{ACA}_0)$) has ordinal $\varphi(\omega{+}1)0$. Only the sharp bounds separate these theories: $f_{\Sigma^1_1\text{-}\mathsf{DC}_0}(\gamma) = \varphi\gamma 0$ and $f_{\mathsf{p}_1^{\omega}(\mathsf{ACA}_0)}(\gamma) = \varphi\omega\gamma$.

Indeed, the sharp bound f_{T} of T stores relevant proof-theoretic information of the infinitary system $\overset{*}{\mathsf{T}}$, and therefore also of T. In terms of reusability, this is good news. If we manipulate T by means of an operation, we can predict what happens to f_{T}. For instance, $\mathsf{p}_1(\mathsf{T})$ proves the existence of an n-tower of ω-models of T, and its sharp bound is obtained by applying to f_{T} the functional It_1 which iterates functions: $\mathsf{It}_1(f)$ is defined to be a normal function so that $\mathsf{It}_1(f, \alpha{+}1) = f(\mathsf{It}_1(f, \alpha))$ (we write $\mathsf{It}_1(f, \alpha)$ for $(\mathsf{It}_1(f))(\alpha)$). Similarly, as $\mathsf{p}_2(\breve{\mathsf{T}})$ plus enough transfinite induction proves $\mathsf{p}_1^{\alpha}(\breve{\mathsf{T}})$, it seems plausible that p_2 corresponds to a type-3 functional It_2 that iterates

[3] $\mathsf{inst}(A \land B) := \mathsf{inst}(A) \cup \mathsf{inst}(B)$, $\mathsf{inst}(\forall X A(X)) := \bigcup_{i \in \mathbb{N}} \mathsf{inst}(A(U_i))$ and $\mathsf{inst}(\forall x A(x)) := \bigcup_{s \text{ a closed term}} \mathsf{inst}(A(s))$; else, $\mathsf{inst}(A) := \{A\}$.

functionals, and that a similar correspondence between operations and functionals persists in higher types.

This approach is especially interesting, because most relevant subsystems of second order arithmetic are obtained by applying to ACA_0 an operation that is composed of the basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$. To reiterate, we have e.g. that $\mathsf{p}_2(\mathsf{ACA}_0) \equiv \Sigma^1_1\text{-}\mathsf{DC}_0$, $\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0) \equiv \mathsf{ATR}_0$, $\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0) \equiv \mathsf{ATR}_0{+}\Sigma^1_1\text{-}\mathsf{DC}_0$ and $\mathsf{p}_2^2(\mathsf{ACA}_0) \equiv \Sigma^1_1\text{-}\mathsf{TDC}_0$ (Σ^1_1-transfinite dependent choice, cf. Rüede [19]).

The main result of this thesis can now be summarized as follows: for each operation p_{n+1}, there is a type-$n{+}2$ functional It_{n+1} that iterates normal functions or type-$n{+}1$ functionals, so that essentially the following holds:

> if f_{ACA_0} is a sharp bound of ACA_0, Op is an operation build from basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$, and H is the functional build from the corresponding basic functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$, then $H(f_{\mathsf{ACA}_0})$ is a sharp bound of $\mathsf{Op}(\mathsf{ACA}_0)$.

In particular, we obtain sharp bounds of a large number of subsystems of second order arithmetic whose ordinals are below $|\bigcup_n \mathsf{p}_{n+1}(\mathsf{ACA}_0)| = |\mathsf{ID}_1|$, the Bachmann-Howard ordinal.

Of course, the above outline neglects many details. An obvious point is that in contrast to functionals, there is no apparent difference between application and composition of operations: while we can apply p_2 and p_1 to ACA_0, the functional It_2 is type-3, and thus only $\mathsf{It}_2(\mathsf{It}_1)$ can be applied to the function f_{ACA_0}. This indicates that e.g. $\mathsf{p}_2(\mathsf{ACA}_0)$ should be regarded as $\mathsf{p}_2\mathsf{p}_1(\mathsf{ACA}_0)$ (still Σ^1_1-DC_0). It also suggests that we have to elaborate on what we mean by "H is the functional build from the corresponding basic functionals" in the above formulation of our main result.

Therefore, we introduce names for operations and functionals. The name x of the operation Op_x codes how this operation is constructed by iterated transfinite composition from the basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$, and the same applies to names of functionals. The correspondence between operations and functionals is then produced by a map $x \mapsto x^H$, so that Op_x relates to H_{x^H}.

Our main result can now be summarized more precisely as follows.

Theorem. *For each name x, let $\mathsf{T}_x := \mathsf{Op}_x(\mathsf{ACA}_0)$, and $f_{x^H} := H_{x^H}(f_{\mathsf{ACA}_0})$. Then, $f_{\mathsf{T}_x} = f_{x^H}$.*

The theorem is proved by induction along some well-founded ordering on the underlying names. We show that $f_{\mathsf{T}_x} = f_{x^H}$, assuming that $f_{\mathsf{T}_y} = f_{y^H}$ holds for all names y that are "simpler" than x.

To check that $f_{x^H} \leq f_{\mathsf{T}_x}$, we extend the notion of a provable ordinal to that of a *provable function* as follows:

$$\mathsf{T} \text{ proves } f :\Leftrightarrow\ \mathsf{T} \vdash \mathsf{Wo}_\lhd(\alpha) \land \mathsf{TI}_\lhd(\mathcal{C}_\mathsf{T}, \alpha) \to \mathsf{Wo}_\lhd(f(\alpha)),$$

where $\mathcal{C}_\mathsf{T}$ is a class depending on T. We show that for each name x, "T_x proves f_{x^H}", which then yields, essentially by the Boundedness Lemma, that $f_{x^H}(\gamma) \leq f_{\overset{*}{\mathsf{T}}_x}(\gamma)$. The converse direction, that $f_{\mathsf{T}_x} \leq g_{x^h}$, is checked by showing that for each limit ordinal γ, and each arithmetical sequent Γ, $\overset{*}{\mathsf{T}}_x \,\big|\!\frac{<\gamma}{+}\, \Gamma \Longrightarrow \overset{*}{\mathsf{T}}^\epsilon \,\big|\!\frac{<f_{x^H}(\gamma)}{*}\, \Gamma$.

Finally, we point out that we obtain also the proof-theoretic ordinals of the theories $\mathsf{T}_x + (\mathsf{I}_\mathsf{N})$, where (I_N) claims formula-induction, i.e., for each L_2-formula $A(u)$,

$$(\mathsf{I}_\mathsf{N}) \qquad \forall x[A(0) \land \forall y(A(y) \to A(y{+}1)) \to A(x)].$$

As for each $\alpha < \varepsilon_0$, $\mathsf{T}_x + (\mathsf{I}_\mathsf{N}) \vdash \mathsf{Wo}_\lhd(\alpha) \land \mathsf{TI}_\lhd(\mathcal{C}_{\mathsf{T}_x}, \alpha)$, we see that "$\mathsf{T}_x$ proves f_{x^H}" implies $\mathsf{T}_x + (\mathsf{I}_\mathsf{N}) \vdash \mathsf{Wo}_\lhd(f_{x^H}(\alpha))$. Therefore, $|\mathsf{T}_x + (\mathsf{I}_\mathsf{N})| \geq f_{\mathsf{T}_x}(\varepsilon_0)$. On the other hand, $\mathsf{T}_x + (\mathsf{I}_\mathsf{N}) \vdash \Gamma$ entails, using standard cut-elimination techniques, that $\overset{*}{\mathsf{T}}_x \,\big|\!\frac{<\varepsilon_0}{*}\, \Gamma$. Therefore $|\mathsf{T}_x + (\mathsf{I}_\mathsf{N})| \leq f_{\mathsf{T}_x}(\varepsilon_0)$.

To conclude this exposition, we list some immediate consequences of the above theorem. To denote the respective proof-theoretic ordinals, we let $\Omega_0 := 1$, $\Omega_{n+1} := \Omega^{\Omega_n}$, $\Omega_0(\alpha) := \alpha$, and $\Omega_{n+1}(\alpha) := \Omega^{\Omega_n(\alpha)}$.

Examples.

(i) $|\mathsf{ACA}_0| = \varepsilon_0$ *and* $|\mathsf{ACA}| = \varphi 1 \varepsilon_0$.

(ii) $|\mathsf{p}_1(\mathsf{ACA}_0)| = \varphi 20$ *and* $|\mathsf{p}_1(\mathsf{ACA}_0) + (\mathsf{I}_\mathsf{N})| = \varphi 2 \varepsilon_0$.

(iii) $|\Sigma^1_1\text{-}\mathsf{DC}_0| = \varphi\omega 0$ *and* $|\Sigma^1_1\text{-}\mathsf{DC}| = \varphi\varepsilon_0 0$.

(iv) $|\mathsf{ATR}_0| = \Gamma_0$ *(Feferman-Schütte ordinal), and* $|\mathsf{ATR}| = \varphi 10\varepsilon_0$.

(v) $|\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi 1\omega 0$ *and* $|\mathsf{ATR} + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi 1 \varepsilon_0 0$.

(vi) $|\Sigma^1_1\text{-}\mathsf{TDC}_0| = \varphi\omega 00$ *and* $|\Sigma^1_1\text{-}\mathsf{TDC}| = \varphi\varepsilon_0 00$.

(vii) $|\mathsf{p}_1(\Sigma^1_1\text{-}\mathsf{TDC}_0)| = \varphi 1000$ *(Ackermann ordinal).*

(viii) $|\mathsf{p}_2^{n+2}(\mathsf{ACA}_0)| = \varphi\omega\underbrace{0\ldots0}_{n}0$, $|\mathsf{p}_2^{n+2}(\mathsf{ACA}_0) + (\mathsf{I}_\mathsf{N})| = \varphi\varepsilon_0\underbrace{0\ldots0}_{n}0$ *and*

$|\mathsf{p}_1\mathsf{p}_2^{n+2}(\mathsf{ACA}_0)| = \varphi 1\underbrace{0\ldots0}_{n+1}0.$

(ix) $|\mathsf{p}_3(\mathsf{ACA}_0)| = \vartheta\Omega^\omega$ *(small Veblen number), and*

$|\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA}_0)| = \vartheta\Omega^\Omega$ *(big Veblen number).*

(x) $|\mathsf{p}_{n+3}(\mathsf{ACA}_0)| = \vartheta\Omega_n(\omega)$, $|\mathsf{p}_{n+3}(\mathsf{ACA})| = \vartheta\Omega_n(\varepsilon_0)$ *and* $\mathsf{p}_1|\mathsf{p}_{n+3}(\mathsf{ACA})| = \vartheta\Omega_{n+1}$.

Contents

In the first part of this thesis, we introduce sets of names Q and Q^H for operations and functionals, and show that for each $x \in Q$, the theory $\mathsf{T}_x := \mathsf{Op}_x(\mathsf{ACA}_0)$ proves the function $H_{x^H}(f_{\mathsf{ACA}_0})$. In the second part, we then show that the function $H_{x^H}(f_{\mathsf{ACA}_0})$ is also a bound of T_x, hence a sharp bound. This also implies that $H_{x^H}(f_{\mathsf{ACA}_0})$ is the largest normal function that is provable in T_x.
Part I consists of the chapters I–IV, and Part II of the chapters V–VI, briefly reviewed below. The second part is shorter, partly because most relevant notions have already been introduced in the first part, but also partly because the computation of bounds is simpler, as we do not have to distinguish between what is provable in a theory and what is true in the mathematical universe outside. We work in a meta-theory and assume a reasonable amount of transfinite induction. At the end, however, we make precise which formal theory would allow to formalize the given proofs.

In the first chapter, we say how we present our theories. This leads to the concept of operations on theories. We introduce the basic operations p_{n+1} and collect some elementary, but relevant properties. Then, we explain how to represent operations by $\mathsf{L}_2(\mathsf{P})$-formulas, and formulate a representation theorem for operations. It states, that we can define new operations from operations with known representations by transfinite recursion, and that there is an $\mathsf{L}_2(\mathsf{P})$-formula which represents this new operation. The proof is rather technical and thus only provided in the appendix. It is not required on a first reading. Next, we introduce the functionals It_{n+1} that iterate functions and functionals, and in some sense correspond to the operations p_{n+1}. We conclude with an auxiliary and technical theorem which provides a substitute for transfinite induction, and allows us to prove, for a well-founded relation $\prec$, certain statements of the form $(\forall \alpha \in \mathsf{field}(\prec))A(\alpha)$ in T^ϵ, for instance our main result.

Chapter II explains most of the ideas of our modular ordinal analysis, however, keeps the framework still simple in that we only consider operations build from the operations p_1 and p_2. After introducing names Q_2^H for functionals and names Q_2 for operations, and moreover, approximations and normal forms of names, we can see our modular ordinal analysis a first time at work.

Chapter III parallels chapter two. We now extend the concepts and results to the general case. This time we consider operations build from the operations p_1 and p_{N_0} for arbitrary large N_0's. As the ordering on names is now more complex, a couple of new problems surface, which are solved by providing additional structure for the sets of names Q and Q^H. This also allows us to cope well with higher type behavior of operations and functionals.

Chapter IV deals with ordinal notations, and how to construct them. We introduce a notation system based on the functionals $(H_x : x \in Q^H)$, the idea being that

(x, α) denotes the ordinal $g_x(\alpha)$. Thereby, we use many ideas developed by Setzer (see [25]). Further, we show how the so obtained notation system relates to more standard ones based on the ϑ-function, or with regard to ordinals below the small Veblen number, to a standard notation system based on the $n$ary Veblen functions.

Chapter V now introduces Tait-style systems for the theories $(\mathsf{T}_x : x \in Q)$ and corresponding infinitary Tait-style systems $(\overset{*}{\mathsf{T}}_x : x \in Q)$ that are suitable to deal with cut-elimination. Further, we extend the language by additional relation symbols $(\mathsf{U}_{n+1} : n \in \mathbb{N})$. The corresponding class terms $\{x : \mathsf{U}_{n+1}(x)\}$ are used to axiomatize new theories, build from given ones, by stacking them on top of each other: $\mathsf{T}_1|\mathsf{T}_0$ ("T_1 over T_0") is essentially the theory $\mathsf{T}^\epsilon + \breve{\mathsf{T}}_1 \wedge \exists X(\breve{\mathsf{T}}_0 \restriction X)$, that is, $\mathsf{T}_1|\mathsf{T}_0$ extends T_1 by an axiom asserting that there is an ω-model of T_0. However, it proves convenient to have an explicit class term for the ω-model above T_0, namely $\{x : \mathsf{U}_1(x)\}$. The need to stack theories on top of each other stems from the way we deal with the operation p_1: basically, we reduce, for a suitable Γ, a derivation $\mathsf{p}_1(\mathsf{T}) \vdash \Gamma$ to a derivation $\mathsf{T}| \dots |\mathsf{T} \vdash \Gamma$. Further, we collect some standard results concerning cut-elimination. Moreover, we show how to cheaply eliminate a cut with a formula A by first replacing the derivation $\overset{*}{\mathsf{T}}_x \overset{<\gamma}{\underset{*}{\vdash}} \Gamma, A$ by a derivation $\overset{*}{\mathsf{T}}_x \overset{<\gamma}{\underset{*}{\vdash}} \Gamma, A'$, where A' is equivalent to A, and the cut with A' is easier to eliminate than the cut with A.

Chapter VI is then devoted to the computation of bounds. After looking at finitary and infinitary reduction properties, we define the notion of a bound and show that it dominates, essentially by the Boundedness Lemma, each provable function. Next, we give two different proofs that the function $H_{x^H}(f_{\mathsf{ACA}_0})$ is not only provable in T_x, but also a bound of T_x. A first direct proof exploits the provided reduction properties by proving the claim by transfinite induction, using a suitably defined norm $|(x, \alpha)|$ of a derivation $\overset{*}{\mathsf{T}}_x \overset{\alpha}{\underset{+}{\vdash}} \Gamma$. And the second and more important proof obtains the same result by first showing a stronger one, namely a dual version of the main result of part I, which states that there is not only a strict correspondence between the theories T_x and the functions $H_{x^H}(f_{\mathsf{ACA}_0})$, but more generally, also between the operations Op_x and the functionals H_{x^H}, and even the operations Op_x^{+n} and the functionals $H_{x^H}^{+n}$ of higher types.

Chapter VI ends with a section "Conclusion", where we give another overview of this thesis. Having the relevant notions at hand by then allows for a more accurate discussion of the underlying ideas and concepts.

Finally, this thesis ends with an appendix which mainly contains the rather technical proof of the so-called Representation Theorem. This theorem states that if we can represent operations $(\mathsf{Op}_i : i \in I)$ within T^ϵ, then also all operations that are obtained by iterated transfinite composition of these operations. This allows to meaningfully talk within formal theories about all operations under consideration.

Part I

Provable Functions

Chapter I

Operations and functionals

In this chapter, we provide the basic concepts required to start a modular computation of provable functions of subsystems of second order arithmetic. After fixing the language and the general form used to present theories in Section 1, we start Section 2 by presenting a general notion of operations on theories. A family $(\mathsf{p}_{n+1} : n \in \mathbb{N})$ of basic operations is defined, out of whose members all operations under consideration are built. Furthermore, we describe how to represent operations within formal theories. Section 3 introduces functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$, which, in some sense made precise later, correspond to the operations p_{n+1}. In Section 4, we prove a technical result which allows us to show already in ACA_0 (and weaker theories) certain statements of a specific form by transfinite induction. This is in particular relevant as our main result of the first part is of such a form.

I.1 Theories

We consider subsystems of second order arithmetic formulated in the language L_2, which comprises the symbol $\in$, a unary relation symbol U, and symbols for the primitive recursive functions and relations. The *number terms* of L_2, denoted by $r, s, t, \ldots$, are defined as usual. For each relation symbol $R(\vec{u})$, each set variable U and all number terms $\vec{s}$ and t, $R(\vec{s})$ and $t \in U$ are atoms of L_2. If A is an atom, then A and $\sim A$ are literals. We write $t \notin U$ for $\sim(t \in U)$. The formulas of L_2 are build from literals by closing under conjunction, disjunction, existential and universal quantification in both sorts. The negation $\neg A$ is defined using De Morgan's laws and the law of double negation. The remaining logical connectives are abbreviated in the standard way. $\mathsf{FV}_0(A)$ denotes the number variables which occur free in A, $\mathsf{FV}_1(A)$ the set variables which occur free in A, and $\mathsf{FV}(A) := \mathsf{FV}_0(A) \cup \mathsf{FV}_1(A)$. $\mathsf{BV}_0(A)$, $\mathsf{BV}_1(A)$ and $\mathsf{BV}(A)$ denote the corresponding sets of variables which occur bound in A. $\mathcal{Q}$ either stands for $\forall$ or $\exists$, and finally, $\top := 0 = 0$ and $\bot := 0 \neq 0$.

An L_2-formula A with $\mathsf{FV}(A) = \emptyset$ is called a sentence, and if $\mathsf{FV}_1(A) = \emptyset$, then A is called an open sentence. Further, a formula without bound set variables is called arithmetical formula, or alternatively Π^1_0-formula or Σ^1_0-formula. Further, A is a Π^1_{n+1}-formula [Σ^1_{n+1}-formula], if A is a Π^1_n-formula or a Σ^1_n-formula or of the form $\forall X B(X)$ [$\exists X B(X)$], where B is a Σ^1_n-formula [Π^1_n-formula]. If $\vec{\mathcal{X}} = \mathcal{X}_1, \dots, \mathcal{X}_n$ and $\vec{\mathcal{Y}} = \mathcal{Y}_1, \dots, \mathcal{Y}_n$ are finite lists of expressions, then $A[\vec{\mathcal{Y}}/\vec{\mathcal{X}}]$ denotes the formula obtained from A by substituting simultaneously all occurrences of the expressions $\vec{\mathcal{X}}$ by $\vec{\mathcal{Y}}$. Further, if a formula is introduced as $A(U, u)$, then $A(X, x)$ is short for $A(U, u)[X, x/U, u]$.

We start by describing the general form of the theories that we consider in this thesis.

Definition I.1.1. *T^ϵ is the L_2-theory that comprises the logical axioms for classical two-sorted predicate calculus, and axioms for the primitive recursive functions and relations. If $\breve{\mathsf{T}}$ is an L_2-sentence, then T denotes the theory $\mathsf{T} := \mathsf{T}^\epsilon + \breve{\mathsf{T}}$ which extends T^ϵ by the single axiom $\breve{\mathsf{T}}$.*

Next, we introduce class terms $\mathcal{C}$ which we also use to code families of sets and classes in the sense specified below.

Definition I.1.2 (Class terms)**.** *Each set variable is a class term, and if $C(\vec{U}, u)$ is an L_2-formula and $\vec{\mathcal{D}}$ are class terms, then $\mathcal{C} := \{x : C(\vec{\mathcal{D}}, x)\}$ is a class term. If $\mathcal{C}$ is the set variable X, then $x \in \mathcal{C}$ is $x \in X$, and if $\mathcal{C}$ is of the form $\{x : C(\vec{\mathcal{D}}, x)\}$, then $x \in \mathcal{C}$ abbreviates $C(\vec{\mathcal{D}}, x)$.*

For each n, we have a standard primitive recursive sequence coding $\langle x_0, \dots, x_{n-1}\rangle$ with associated projections $(\langle x_0, \dots, x_{n-1}\rangle)_i = x_i$ for $0 \leq i < n$. Also, we often regard a class $\mathcal{C}$ as a family $\{(\mathcal{C})_n : n \in \mathbb{N}\}$ of classes, where $(\mathcal{C})_t := \{x : \langle x, t\rangle \in \mathcal{C}\}$. Moreover, $(\mathcal{C})_{s,t}$ is short for $((\mathcal{C})_s)_t$ and $\mathcal{C} = \mathcal{C}'$ abbreviates that $\mathcal{C}$ and $\mathcal{C}'$ have the same elements. Finally, $\emptyset$ denotes the class term $\{x : x \neq x\}$ and $\mathsf{N} := \{x : x = x\}$.

Definition I.1.3. *The following notations allow to restrict the range of bound set variables to the classes coded by a class term $\mathcal{C}$.*

(i) For each L_2-formula $A(U)$, $(\mathcal{Q}X \dot{\in} \mathcal{C})A(X) := \mathcal{Q}x A((\mathcal{C})_x)$, where x is a fresh variable.

(ii) $A{\restriction}\mathcal{C} := A$ if A is arithmetical; else $(A\, j\, B){\restriction}\mathcal{C} := A{\restriction}\mathcal{C}\, j\, B{\restriction}\mathcal{C}$ for $j \in \{\wedge, \vee\}$, $(\mathcal{Q}xB){\restriction}\mathcal{C} := \mathcal{Q}x(B{\restriction}\mathcal{C})$ and $(\mathcal{Q}X A(X)){\restriction}\mathcal{C} := (\mathcal{Q}X \dot{\in} \mathcal{C})(A(X){\restriction}\mathcal{C})$.

Further, $X \dot{\in} \mathcal{C} := \exists x[X = (\mathcal{C})_x]$. Moreover, ${\restriction}$ takes precedence over quantifiers and logical connectives: for $j \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, $A\, j\, B{\restriction}\mathcal{C} := A\, j\, (B{\restriction}\mathcal{C})$, $\mathcal{Q}X A{\restriction}\mathcal{C} := \mathcal{Q}X(A{\restriction}\mathcal{C})$ and $\mathcal{Q}x A{\restriction}\mathcal{C} := \mathcal{Q}x(A{\restriction}\mathcal{C})$.

If $\mathcal{C}$ and $\mathcal{D}$ are class terms, then $\mathcal{C} = \{x : C(x)\}$ for some formula $C(u)$, thus $\mathcal{C}\restriction\mathcal{D}$ is $\{x : C(x)\restriction\mathcal{D}\}$. Occasionally, we also write $A^{\mathcal{C}}$ for $A\restriction\mathcal{C}$.

Remark I.1.4. *Let us address a possible source of confusion. Note that if e.g. $A(U)$ is arithmetical, then $(\forall X\dot{\in}\mathcal{C})A(X)$ is $\forall x A((\mathcal{C})_x)$, where in general $(\mathcal{C})_x$ is not a set. Hence $\forall x A((\mathcal{C})_x)$ implies $\forall X(X\dot{\in}\mathcal{C} \to A(X))$, but the latter formula claims $A(X)$ only for sets X of the form $X = (\mathcal{C})_x$ for some x.*

Observe that in $A\restriction\mathcal{C}$ only the range of the bound set variables of A is restricted. Below, we define the formula $\mathcal{C} \models A$, which also restricts the extension of the free set variables in $\mathsf{FV}_1(A) \setminus \mathsf{FV}_1(\mathcal{C})$.

Definition I.1.5. *Let $\mathcal{C}$ be a class term and A an L_2-formula with $\mathsf{FV}_1(A)\setminus\mathsf{FV}_1(\mathcal{C}) = \{U_1, \ldots, U_n\}$. Then,*

$$\mathcal{C} \models A := (A\restriction\mathcal{C})[(\mathcal{C})_{v_1}/U_1, \ldots, (\mathcal{C})_{v_n}/U_n],$$

where $v_1, \ldots, v_n$ are pairwise distinct fresh number variables (i.e. variables that do not occur in $A\restriction\mathcal{C}$). To be specific, assume that $v_1, \ldots, v_n$ are the first variables w.r.t. some fixed enumeration that do no occur in $A\restriction\mathcal{C}$.

Note that if $\mathsf{FV}_1(A) \setminus \mathsf{FV}_1(\mathcal{C}) = \emptyset$, then $A\restriction\mathcal{C}$ is the same formula as $\mathcal{C} \models A$. We continue by recording some properties concerning the abbreviations $X\dot{\in}\mathcal{C}$, $A\restriction\mathcal{C}$ and $X \models A$.

Lemma I.1.6. T^{ϵ} *proves the following:*

(i) $\exists X[X\dot{\in}\mathcal{C} \land A(X)] \to (\exists X\dot{\in}\mathcal{C})A(X)$.

(ii) $(\forall X\dot{\in}\mathcal{C})A(X) \to \forall X[X\dot{\in}\mathcal{C} \to A(X)]$.

Proof We work informally in T^{ϵ}. (i) If there is an X with $X\dot{\in}\mathcal{C} \to A(X)$, then for some x, $X = (\mathcal{C})_x$, and so also $A((\mathcal{C})_x)$, that is, $\exists x A((\mathcal{C})_x)$, which is $(\exists X\dot{\in}\mathcal{C})A(X)$. (ii) Assume that x does not occur in $A(\mathcal{C})$. Then $(\forall X\dot{\in}\mathcal{C})A(X)$ is $\forall x A((\mathcal{C})_x)$. Hence for each x, $A((\mathcal{C})_x)$, in particular $A(X)$ in case that $X = (\mathcal{C})_y$ for some y, so $\exists y(X = (\mathcal{C})_y) \to A(X)$, that is $X\dot{\in}\mathcal{C} \to A(X)$, and $\forall X[X\dot{\in}\mathcal{C} \to A(X)]$ follows. $\square$

There is a caveat though: $\exists X[X\dot{\in}\mathcal{C} \land A(X)]$ claims the existence of a set, whereas $(\exists X\dot{\in}\mathcal{C})A(X)$ only claims $A((\mathcal{C})_x)$ for some class $(\mathcal{C})_x$. The two statements are only equivalent if we have $\forall x\exists Y[Y = (\mathcal{C})_x]$, stating that each class $(\mathcal{C})_x$ is a set.

Lemma I.1.7. T^{ϵ} *proves the following: if $\forall x\exists Y[Y = (\mathcal{C})_x]$, then*

(i) $\exists X[X\dot{\in}\mathcal{C} \land A(X)] \leftrightarrow (\exists X\dot{\in}\mathcal{C})A(X)$.

(ii) $\forall X[X\dot{\in}\mathcal{C} \to A(X)] \leftrightarrow (\forall X\dot{\in}\mathcal{C})A(X)$.

Next, we state two basic observations concerning class terms that are tacitly used later. A simple direct proof of these claims is obtained analogously to the proof of Lemma V.2.8 in the second part of this thesis. Below, just a model theoretic arguments is given.

Definition I.1.8. *A structure $\mathbb{M} = (\mathcal{N}, \mathbb{S}, \mathcal{U})$ for the language L_2 consists of a structure $\mathcal{N} = (\mathsf{N}, \ldots)$ for the first order part of L_2, a non-empty collection $\mathbb{S}$ of subsets of N used to interpret the set variables, and a set $\mathcal{U} \subseteq \mathsf{N}$ to interpret the relation symbol U. If $\mathbb{M}$ satisfies the axioms of T, $\mathbb{M}$ is called a model of T. Given a structure $\mathbb{M}$, $\mathcal{V}$ denotes a valuation that maps number variables to $\mathcal{N}$ and set variables to $\mathbb{S}$. If A is a formula with free variables, then as usual, $\mathbb{M}_{\mathcal{V}} \models A$ states that $\mathbb{M}$ satisfies A under the variable assignment $\mathcal{V}$.*

Lemma I.1.9. *For each open L_2-sentence A and each class term $\mathcal{C}$,*

$$\mathsf{T}^\epsilon \vdash A \Rightarrow \mathsf{T}^\epsilon \vdash A{\restriction}\mathcal{C}.$$

Proof Assume that $\mathcal{C} = \mathcal{C}(\vec{V})$, and that $\mathsf{T}^\epsilon \vdash A$. We show that $\mathsf{T}^\epsilon \nvdash A{\restriction}\mathcal{C}$ is impossible. If this were the case, then there exists a countable model $\mathbb{M} = (\mathcal{N}, \mathbb{S}, \mathcal{U})$ of T^ϵ, sets $\vec{\mathcal{Z}} \in \mathbb{S}$ and a valuation $\mathcal{V}$ of the number variables, so that $\mathbb{M}_{\mathcal{V}} \not\models A{\restriction}\mathcal{C}(\vec{\mathcal{Z}})$. Now let $\mathbb{S}' := \{\mathcal{X} : \mathbb{M}_{\mathcal{V}} \models (\exists X \dot{\in} \mathcal{C}(\vec{\mathcal{Z}}))(X = \mathcal{X})\}$ be the collection of sets coded by the interpretation of the class term $\mathcal{C}$, that is, the range of the quantified set variables of A. As T^ϵ contains no axioms for sets and $\mathbb{S}'$ is not empty, $\mathbb{M}' := (\mathcal{N}, \mathbb{S}', \mathcal{U})$ is a model of T^ϵ. By choice of $\mathbb{S}'$ and $\mathcal{V}$, $\mathbb{M}'_{\mathcal{V}} \not\models A$, contradicting $\mathsf{T}^\epsilon \vdash A$. □

The following example shows the reason for the restriction to open L_2-sentences: $\mathsf{T}^\epsilon \vdash \exists X[X = U]$, but since $\mathsf{T}^\epsilon \nvdash \emptyset = U$, also $\mathsf{T}^\epsilon \nvdash (\exists X \dot{\in} \emptyset)[X = U]$.

The converse direction fails in general. For instance, let $A := \exists X(X = \emptyset)$ and $\mathcal{C} := \emptyset$. Note that for each x, $(\emptyset)_x = \emptyset$ and that $A{\restriction}\mathcal{C} = \exists x((\emptyset)_x = \emptyset)$. Hence $\mathsf{T}^\epsilon \vdash A{\restriction}\mathcal{C}$, but $\mathsf{T}^\epsilon \nvdash A$.

Lemma I.1.10. *For each L_2-formula A with $\mathsf{FV}_1(A) = \{U_1, \ldots, U_n\}$ and each class term $\mathcal{C}$ with $\mathsf{FV}_1(A) \cap \mathsf{FV}_1(\mathcal{C}) = \emptyset$, we have that (i)⇒(ii)⇒(iii), where*

(i) $\mathsf{T}^\epsilon \vdash A$,

(ii) $\mathsf{T}^\epsilon \vdash \mathcal{C} \models A$,

(iii) $\mathsf{T}^\epsilon \vdash \vec{U} \dot{\in} \mathcal{C} \to A{\restriction}\mathcal{C}$.

Proof If $\mathsf{T}^\epsilon \vdash A(\vec{U})$, then $\mathsf{T}^\epsilon \vdash \forall \vec{X} A(\vec{X})$, so $\mathsf{T}^\epsilon \vdash \forall x_1, \ldots, x_n A((\mathcal{C})_{x_1}, \ldots, (\mathcal{C})_{x_n}){\restriction}\mathcal{C}$ by Lemma I.1.9, and thus $\mathsf{T}^\epsilon \vdash A((\mathcal{C})_{v_1}, \ldots, (\mathcal{C})_{v_n}){\restriction}\mathcal{C}$ for fresh variables $\vec{v}$, that is, $\mathsf{T}^\epsilon \vdash \mathcal{C} \models A$. This clearly entails $\mathsf{T}^\epsilon \vdash \vec{U} \dot{\in} \mathcal{C} \to A{\restriction}\mathcal{C}$. □

To round the picture, we also show that the statements in the above lemma are actually all equivalent in case that the class term $\mathcal{C}$ is a set variable. We state this fact as a separate lemma, since it is not required in the sequel, and since we just give a model theoretic proof.

Lemma I.1.11. *For each* L_2*-formula* A *with* $Z \notin \mathsf{FV}_1(A) = \{U_1, \ldots, U_n\}$*, we have that* $\mathsf{T}^\epsilon \vdash \vec{U}\dot{\in}Z \to A{\restriction}Z$ *implies* $\mathsf{T}^\epsilon \vdash A$*.*

Proof If $\mathsf{T}^\epsilon \nvdash A$, then there is a countable model $\mathbb{M} = (\mathcal{N}, \mathbb{S}, \mathcal{U})$ of T^ϵ, and sets $\vec{\mathcal{Y}} \in \mathbb{S}$ so that $\mathbb{M} \models \neg A(\vec{\mathcal{Y}})$. Let $(\mathcal{X}_i : i \in \mathsf{N})$ be an enumeration of the sets in $\mathbb{S}$, $\mathcal{Z} := \{\langle x, i\rangle : x \in \mathcal{X}_i\}$, $\mathbb{S}' = \mathbb{S} \cup \{\mathcal{Z}\}$ and $\mathbb{M}' = (\mathsf{N}, \mathbb{S}', \mathcal{U})$. By choice of $\mathcal{Z}$, $\mathbb{M}'$ is a model of T^ϵ with $\mathbb{M}' \models \vec{\mathcal{Y}}\dot{\in}\mathcal{Z} \wedge \neg A(\vec{\mathcal{Y}}){\restriction}\mathcal{Z}$, contradicting $\mathsf{T}^\epsilon \vdash \vec{U}\dot{\in}Z \to A{\restriction}Z$. □

A typical application is the following.

Corollary I.1.12. *If* $\mathsf{FV}(\breve{\mathsf{T}}) = \emptyset$*, then* $\mathsf{T}^\epsilon + \breve{\mathsf{T}} \vdash A$ *iff* $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}{\restriction}X \to X \models A$*.*

Proof $\mathsf{T}^\epsilon + \breve{\mathsf{T}} \vdash A$ iff $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to A$ iff $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}{\restriction}X \to X \models A$. □

We conclude this section by giving the axioms of three important subsystems of second order arithmetic, ACA_0, Σ^1_1-DC_0 and ATR_0. ACA_0 is the weakest system which we consider. The theories Σ^1_1-DC_0 and ATR_0 can be obtained by applying certain operations to ACA_0, but are also of independent interest.

In order to formulate the axioms of ATR_0, we introduce the abbreviation

$$(X)_{\prec t} := \{\langle x, y\rangle \in X : y \prec t\},$$

and some further abbreviations which are also extensively used when dealing with provable functions. For each binary relation symbol $\prec$, $\mathsf{Wf}_\prec(u) := \forall X \mathsf{TI}_\prec(X, u)$ asserts that $\prec{\restriction}u :=\prec{\restriction}\{y : y \prec u\}$ is well-founded, where

$$\begin{aligned} \mathsf{Prog}_\prec(U) &:= (\forall x \in \mathsf{field}(\prec))((\forall y \prec x)(y \in U) \to (x \in U)), \\ \mathsf{TI}_\prec(U, u) &:= \mathsf{Prog}_\prec(U) \to (\forall y \prec u)(y \in U). \end{aligned}$$

$\mathsf{Wo}_\prec(u)$ claims that $\mathsf{Wf}_\prec(u)$ and that $\prec{\restriction}\{y : y \preceq u\}$ is a strict linear order. Further, we use the above abbreviations with a set variable in place of the relation symbol $\prec$. In this case, xUy is read as $\langle x, y\rangle \in U$. Often, we also use $\prec$ as a set variable. Moreover, $\mathsf{Wo}(U) := \forall x \mathsf{Wo}_U(x)$ and $\mathsf{Wf}(U) := \forall x \mathsf{Wf}_U(x)$.

The aforementioned theories as well as most second order theories with set induction can be presented in the form $\mathsf{T}^\epsilon + \breve{\mathsf{T}}$. In this case, the axiom $\breve{\mathsf{T}}$ asserts the existence of certain sets, and further, implies the sentence $\forall X \mathsf{IND}(X)$ claiming set induction, where

$$\mathsf{IND}(U) := \forall x[0 \in U \wedge \forall y(y \in U \to y{+}1 \in U) \to x \in U].$$

Occasionally, also the schema of formula induction is considered, which claims for each L_2-formula $A(u)$,

$$(\mathsf{I}_\mathsf{N}) \qquad \forall x[A(0) \land \forall y(A(y) \to A(y+1)) \to A(x)].$$

Definition I.1.13. *The theories* ACA_0, Σ^1_1-DC_0 *and* ATR_0 *are defined as follows.*

(i) ACA_0 *extends* T^ϵ *by set induction, i.e.* $\forall X \mathsf{IND}(X)$*, and an axiom*

$$\exists Z[Z = \{z : A(z)\}].$$

for each arithmetical formulas $A(u)$*.*

(ii) Σ^1_1-DC_0 *extends* ACA_0 *by an axiom*

$$\forall X \exists Y A(X,Y) \to \exists Z[W = (Z)_0 \land \forall n A((Z)_n, (Z)_{n+1})]$$

for each arithmetical formula $A(U,V)$*.*

(iii) ATR_0 *extends* ACA_0 *by an axiom*

$$\mathsf{Wo}(\prec) \to \exists F \forall y((F)_y = \{x : A((F)_{\prec y}, x)\}),$$

for each arithmetical formula $A(U,u)$*.*

The following is well-known (a specific Π^1_2-sentence that axiomatizes ACA_0 is given in Section 2 of the appendix).

Lemma I.1.14. *There is a* Π^1_2*-sentence* (ACA)*, so that* ACA_0 *is equivalent to* $\mathsf{T}^\epsilon + (\mathsf{ACA})$*.*

To forestall future confusion, we stress that (ACA) is a L_2-sentences so that $\mathsf{T}^\epsilon + (\mathsf{ACA})$ is ACA_0. In particular, (ACA) is not to be confused with ACA, often used in the literature to denote the theory $\mathsf{ACA}_0 + (\mathsf{I}_\mathsf{N})$.

I.2 Operation

Our focus on theories of the form $\mathsf{T} := \mathsf{T}^\epsilon + \breve{\mathsf{T}}$ allows us to regard operations on theories as maps on L_2-sentences. However, to deal also with internalized versions of theories, we define operations to be maps on open sentences, instead. Say, we have a family $(\mathsf{T}_n : n \in \mathbb{N})$ of theories. To prove that for each n, $\mathsf{T}_n \vdash A_n$, it might be worth attempting to prove an internal variant $\mathsf{T}^\epsilon \vdash \forall x[\breve{\mathsf{T}}(x) \to A(x)]$, where $\breve{\mathsf{T}}(u)$ and $A(u)$ are open sentences with $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}(\overline{n}) \leftrightarrow \breve{\mathsf{T}}_n$ and $\mathsf{T}^\epsilon \vdash A(\overline{n}) \leftrightarrow A_n$. And in the course of such an argument, we may want to apply an operation to $\breve{\mathsf{T}}(u)$.

This is how operations are introduced in the subsection below. In the next subsection, we then show how to represent operations as $\mathsf{L}_2(\mathsf{P})$-formulas (formulas with an additional relation symbol $\mathsf{P}(U)$, cf. Definition I.2.16), and how to define transfinite iterations of operations.

I.2.1 Operations as functions on open L_2-sentences

Essentially, an operation Op maps a theory $\mathsf{T}^\epsilon + \breve{\mathsf{T}}$ to the theory $\mathsf{T}^\epsilon + \mathsf{Op}(\breve{\mathsf{T}})$. For reasons discussed below, we impose some further conditions on the map Op.

Definition I.2.1. *An operation* Op *is a function that maps an open* L_2*-sentence to an open* L_2*-sentence, so that for all open* L_2*-sentence* $\breve{\mathsf{T}}$ *and* $\breve{\mathsf{T}}'$*,*

(i) $\mathsf{FV}_0(\breve{\mathsf{T}}) \subseteq \mathsf{FV}_0(\mathsf{Op}(\breve{\mathsf{T}}))$*,*

(ii) $\mathsf{T}^\epsilon \vdash \forall X(\breve{\mathsf{T}}{\restriction}X \to \breve{\mathsf{T}}'{\restriction}X) \to (\mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}'))$*.*

If a theory is introduced as $\mathsf{T} := \mathsf{T}^\epsilon + \breve{\mathsf{T}}$*, then* $\mathsf{Op}(\mathsf{T}) := \mathsf{T}^\epsilon + \mathsf{Op}(\breve{\mathsf{T}})$*.* $\mathsf{Op}^0(\mathsf{T}) := \mathsf{T}$ *and* $\mathsf{Op}^{n+1}(\mathsf{T}) := \mathsf{Op}(\mathsf{Op}^n(\mathsf{T}))$*. Further, if* Op *and* Op' *are operations,* $(\mathsf{Op} \circ \mathsf{Op}')(\breve{\mathsf{T}}) := \mathsf{Op}(\mathsf{Op}'(\breve{\mathsf{T}}))$*.*

Henceforth, $\breve{\mathsf{T}}$ and $\breve{\mathsf{T}}'$ range over open L_2-sentences. Observe that if $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$, then by Lemma I.1.9, $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}{\restriction}X \to \breve{\mathsf{T}}'{\restriction}X$, and thus by (ii) of the above definition, $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}')$. Therefore, (ii) is a stronger condition than $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}' \Rightarrow \mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}')$. On the other hand, although we have for an open sentence A that $\mathsf{T}^\epsilon \vdash A$ implies $\mathsf{T}^\epsilon \vdash A{\restriction}X$, it may of course be the case that $\mathsf{T}^\epsilon \nvdash A \to A{\restriction}X$. Therefore, it is not surprising that there are $\breve{\mathsf{T}}$ and $\breve{\mathsf{T}}'$, so that

$$\mathsf{T}^\epsilon \nvdash (\breve{\mathsf{T}} \to \breve{\mathsf{T}}') \to \forall X(\breve{\mathsf{T}}{\restriction}X \to \breve{\mathsf{T}}'{\restriction}X).$$

In fact, we have even $\mathsf{T}^\epsilon \nvdash (\breve{\mathsf{T}} \to \breve{\mathsf{T}}') \to (\mathsf{p}_1(\breve{\mathsf{T}}) \to \mathsf{p}_1(\breve{\mathsf{T}}'))$, as detailed in Remark I.2.11. Item (ii) tries to approximate $(\breve{\mathsf{T}} \to \breve{\mathsf{T}}') \to (\mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}'))$ by replacing $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$ by "$\breve{\mathsf{T}} \to \breve{\mathsf{T}}'$ holds in all models".

Definition I.2.2. *If* $\mathsf{FV}_0(\mathsf{Op}(\breve{\mathsf{T}})) \setminus \mathsf{FV}_0(\breve{\mathsf{T}}) = \{u_1, \ldots, u_n\}$*, then we may highlight this by writing* $\mathsf{Op}_{\vec{u}}$ *for* Op*. In this case,* $\mathsf{Op}_{\vec{t}}(\breve{\mathsf{T}}) := \mathsf{Op}(\breve{\mathsf{T}})[\vec{t}/\vec{u}]$*.*

All operations we are interested in are build from basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$, which are defined below (cf. Definition I.2.8). Informally speaking, $\mathsf{p}_1(\mathsf{T})$ claims the existence of arbitrary large models of T: for each set Z, there is an X with $Z \dot{\in} X$ so that $\breve{\mathsf{T}}{\restriction}X$. And $\mathsf{p}_{n+2}(\mathsf{T})$ claims that there are arbitrary large models of T that are Π^1_{n+2}-reflecting: for each Π^1_{n+2}-formula $A(U)$ and each set Z, if $A(Z)$, then there is an X with $Z \dot{\in} X$, so that $\breve{\mathsf{T}}{\restriction}X$ and $A(Z){\restriction}X$.

When working in $\mathsf{p}_{n+1}(\mathsf{T})$, given sets Z_1 and Z_2, we also want models X of T that contain Z_1 and Z_2. Therefore, we state that the theories $\mathsf{p}_{n+1}(\mathsf{T})$ meet the axiom pair, which allows to add a set to a family of sets; for each X and Y there is a set $Z := X{+}Y$ so that $(Z)_0 = X$ and $(Z)_{n+1} = (Y)_n$ for each n. Further, we need that with X also $(X)_x$ is a set, which is ensured by the axiom trans.

Definition I.2.3. *The Π^1_2-sentences* trans *and* pair *state the following:*

(i) $\mathsf{trans} := \forall X, x \exists Y[Y = (X)_x]$.

(ii) $\mathsf{pair} := \forall X, Y \exists Z[Z = X{+}Y]$, *where*
$X{+}Y := \{\langle x, 0\rangle : x \in X\} \cup \{\langle y, z{+}1\rangle : \langle y, z\rangle \in Y\}$.

To keep subsequent definitions as simple as possible, we apply operations only to theories that imply (ACA), and henceforth stick to the following convention.

Convention I.2.4. *Unless stated otherwise, an anonymous* $\breve{\mathsf{T}}$ *is always assumed to be an open* L_2*-sentences so that* $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to (\mathsf{ACA})$*, and an anonymous theory* T *is always assumed to be of the form* $\mathsf{T}^\epsilon + \breve{\mathsf{T}}$*. This assumption is justified by Remark I.2.13.*

Now we say that A is Π^1_n, if there is a Π^1_n-formula B with $\mathsf{FV}(A) = \mathsf{FV}(B)$ so that $\mathsf{T}^\epsilon \vdash A \leftrightarrow B$. Note that by the above convention, we have that $\mathsf{ACA}_0 \vdash \breve{\mathsf{T}} \leftrightarrow \breve{\mathsf{T}}'$ iff $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \leftrightarrow \breve{\mathsf{T}}'$.

Having fixed a domain of theories, we read $\mathsf{Op} \Leftrightarrow \mathsf{Op}'$ and $\mathsf{Op} \Rightarrow \mathsf{Op}'$ in the following way.

Definition I.2.5. *We write* $\mathsf{Op} \Leftrightarrow \mathsf{Op}'$ *(and also* Op *iff* Op'*), if for each* $\breve{\mathsf{T}}$ *(which implies* (ACA)*),* $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \leftrightarrow \mathsf{Op}'(\breve{\mathsf{T}})$*. Accordingly,* $\mathsf{Op} \Rightarrow \mathsf{Op}'$ *expresses that for each* $\breve{\mathsf{T}}$*,* $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}'(\breve{\mathsf{T}})$*.*

Lemma I.2.6. *If* $\mathsf{Op}' \Rightarrow \mathsf{Op}''$*, then* $\mathsf{Op} \circ \mathsf{Op}' \Rightarrow \mathsf{Op} \circ \mathsf{Op}''$*.*

Proof Fix some $\breve{\mathsf{T}}$ and assume $\mathsf{Op}' \Rightarrow \mathsf{Op}''$. Then, $\mathsf{T}^\epsilon \vdash \mathsf{Op}'(\breve{\mathsf{T}}) \to \mathsf{Op}''(\breve{\mathsf{T}})$, hence also $\mathsf{T}^\epsilon \vdash \forall X[\mathsf{Op}'(\breve{\mathsf{T}}){\restriction}X \to \mathsf{Op}''(\breve{\mathsf{T}}){\restriction}X$ by Lemma I.1.9. As Op is an operation, we have that $\mathsf{T}^\epsilon \vdash (\mathsf{Op} \circ \mathsf{Op}')(\breve{\mathsf{T}}) \to (\mathsf{Op} \circ \mathsf{Op}'')(\breve{\mathsf{T}})$ follows. □

The definition of the operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$ makes use of so-called universal formulas.

Definition I.2.7 (Universal formulas)**.** *For each $n > 0$, we denote by $\pi^1_n(U, u, e)$ a Π^1_n-formula with the property that for each each Π^1_n-formula $A(U, u, v)$,*

$$\mathsf{ACA}_0 \vdash \forall y\, \exists e \forall X, x[A(X, y, x) \leftrightarrow \pi^1_n(X, x, e)].$$

Moreover, the index e in the universal formula π^1_n is effectively computable from the formula A and the number parameter y.

Details about universal formulas can be found e.g. in Simpson [26] (cf. Definition VII.1.3), and Probst [12] (Corollary II.1.12).

Definition I.2.8 (The basic operations). *For each natural number n, we define $R_{n+1}(X,Z,x,e) := \pi^1_{n+1}(Z,x,e) \to \pi^1_{n+1}(Z,x,e){\restriction}X$. Then,*

(i) $\mathsf{p}_1(\breve{\mathsf{T}}) := \forall Z \exists X (Z \dot{\in} X \wedge \breve{\mathsf{T}}{\restriction}X) \wedge \mathsf{pair} \wedge \mathsf{trans}$, *and*

(ii) $\mathsf{p}_{n+2}(\breve{\mathsf{T}}) := \forall Z \forall x, e \exists X [Z \dot{\in} X \wedge \breve{\mathsf{T}}{\restriction}X \wedge R_{n+2}(X,Z,x,e)] \wedge \mathsf{pair} \wedge \mathsf{trans}$.

That each p_{n+1} is an operation is immediate by its definition, and we readiy observe that $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ is Π^1_{n+2}, and that $\mathsf{p}_{n+2}(\mathsf{T}) \vdash \mathsf{p}_{n+1}(\breve{\mathsf{T}})$. Further note that $\mathsf{T}^\epsilon + \mathsf{trans} \vdash \pi^1_1(Z,x,e) \to \pi^1_1(Z,x,e){\restriction}X$, hence, over the theory T^ϵ, $\mathsf{p}_1(\breve{\mathsf{T}})$ is equivalent to $\forall Z \forall x, e \exists X [Z \dot{\in} X \wedge \breve{\mathsf{T}}{\restriction}X \wedge R_1(X,Z,x,e)] \wedge \mathsf{pair} \wedge \mathsf{trans}$.

Remark I.2.9. *Note that over ACA_0, $\mathsf{p}_{n+1}((\mathsf{ACA}))$ is equivalent to the ω-model reflection scheme for Σ^1_{n+2}-formulas, introduced in Simpson ([26], Definition VIII.5.1): for each Σ^1_{n+2}-formula $A(\vec{V})$ with $\mathsf{FV}(A) = \{V_1, \ldots, V_k\}$,*

$$(\Sigma^1_{n+2}\text{-}\mathsf{RFN}) \qquad \forall \vec{Z}[A(\vec{Z}) \to \exists X[\vec{Z} \dot{\in} X \wedge (\mathsf{ACA}){\restriction}X \wedge A(\vec{Z}){\restriction}X].$$

To see that $\mathsf{p}_{n+1}((\mathsf{ACA}))$ implies $(\Sigma^1_{n+2}\text{-}\mathsf{RFN})$, let $A(\vec{V}) = \exists W B(W,\vec{V})$, where B is Π^1_{n+1}, and assume that $\exists W B(W,\vec{Y})$. Let Z so that $(Z)_0 = W$ and $(Z)_{i+1} = Y_i$ for $0 \le i \le k$. For some e, $B(W,\vec{Y})$ iff $\pi^1_{n+1}(Z,e)$, and $\mathsf{p}_{n+1}(\mathsf{ACA})$ provides a set X so that $(\mathsf{ACA}){\restriction}X$ and $Z \dot{\in} X$ (hence also $W, \vec{Y} \dot{\in} X$), and $(\exists W B(W,\vec{Z})){\restriction}X$.
For the converse direction, assume that $\pi^1_{n+1}(Z,x,e)$. Then, there is a Π^1_{n+1}-formula $B(U)$ with $\mathsf{FV}(B) = \{U\}$, so that for $W := \{\langle z, 0\rangle : z \in Z\} \cup \{\langle x, e{+}1\rangle\}$, $B(W)$ iff $\pi^1_{n+1}(Z,x,e)$. Hence, $\exists W B(W)$, and Σ^1_{n+2} ω-model reflection yields a set X, so that $(\mathsf{ACA}){\restriction}X$, and there is a $W \dot{\in} X$ so that $B(W){\restriction}X$. With $W \dot{\in} X$, also $Z \dot{\in} X$, therefore $\pi^1_{n+1}(Z,x,e){\restriction}X$.

Further, we recall the following well-known facts.

Lemma I.2.10. *$\mathsf{p}_2(\mathsf{ACA}_0)$ is Σ^1_1-DC_0 and $\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0)$ is ATR_0.*

Proof This can be found e.g. in Simpson [26]. The first claim is Theorem VIII.5.12. The right-to-left direction of the second claim is by Theorem VIII.4.20, and the direction from left-to-right is by Lemma VIII.4.15 and Theorem VIII.3.15. □

Let (ACA) be the Π^1_2-sentence provided by Lemma I.1.14. So ACA_0 is $\mathsf{T}^\epsilon + (\mathsf{ACA})$, Σ^1_1-DC_0 is $\mathsf{T}^\epsilon + (\Sigma^1_1\text{-}\mathsf{DC})$ for $(\Sigma^1_1\text{-}\mathsf{DC}) := \mathsf{p}_2((\mathsf{ACA}))$, and ATR_0 is $\mathsf{T}^\epsilon + (\mathsf{ATR})$ for $(\mathsf{ATR}) := \mathsf{p}_1((\Sigma^1_1\text{-}\mathsf{DC}))$. Observe that Σ^1_1-DC_0 is Π^1_3, and that (ATR) is Π^1_2.

Remark I.2.11. *As shown in Simpson [26] Theorem VIII.5.13, there exists a model $\mathbb{M}$ of ATR_0 that does neither satisfy $(\Sigma^1_1\text{-}\mathsf{DC})$ nor $\mathsf{p}_1((\mathsf{ATR}))$. Then,*

$$\mathbb{M} \not\models ((\Sigma^1_1\text{-}\mathsf{DC}) \to (\mathsf{ATR})) \to [\mathsf{p}_1((\Sigma^1_1\text{-}\mathsf{DC})) \to \mathsf{p}_1((\mathsf{ATR}))],$$

as trivially, $\mathbb{M} \models (\Sigma^1_1\text{-}\mathsf{DC}) \to (\mathsf{ATR})$ and $\mathbb{M} \models \mathsf{p}_1((\Sigma^1_1\text{-}\mathsf{DC}))$. This illustrates that there are $\breve{\mathsf{T}}$ and $\breve{\mathsf{T}}'$, so that $\mathsf{T}^\epsilon \not\vdash (\breve{\mathsf{T}} \to \breve{\mathsf{T}}') \to (\mathsf{p}_1(\breve{\mathsf{T}}) \to \mathsf{p}_1(\breve{\mathsf{T}}'))$.

Next, we collect some basic properties of the operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$.

Lemma I.2.12. *Assume that* $\breve{\mathsf{T}}'$ *is an open* Π^1_{n+2}*-sentence of* L_2*. If* $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$*, then* $\mathsf{T}^\epsilon \vdash \mathsf{p}_{n+1}(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}'$*.*

Proof We start with a preparatory consideration. Assume that $\breve{\mathsf{T}}' = \forall Z C(Z)$ for some Σ^1_{n+1}-formula $C(U)$, and that further $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$. Then also $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to C(U)$. Now Lemma I.1.10 yields $\mathsf{T}^\epsilon \vdash U\dot{\in}X \to [\breve{\mathsf{T}}{\restriction}X \to C(U){\restriction}X]$, from which we readily obtain

$$(*) \qquad \mathsf{T}^\epsilon \vdash \forall X[U\dot{\in}X \wedge \breve{\mathsf{T}}{\restriction}X \to C(U){\restriction}X].$$

Now we show $\mathsf{T}^\epsilon \vdash \mathsf{p}_{n+1}(\breve{\mathsf{T}}) \to \forall Z C(Z)$ by induction on n. We work informally in T^ϵ, assume that $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ and $\neg C(Z)$ for some Z, and argue for a contradiction. If $n = 0$, $\mathsf{p}_1(\breve{\mathsf{T}})$ provides a set X so that $Z\dot{\in}X$ and $\breve{\mathsf{T}}{\restriction}X$. Now $(*)$ yields $C(Z){\restriction}X$. As C is Σ^1_1 and for each z, $(X)_z$ is a set since $\mathsf{p}_1(\breve{\mathsf{T}})$ implies trans, we obtain $C(Z)$. And if $n > 0$, then $\neg C$ is Π^1_{n+1}, and $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ provides an X so that $Z\dot{\in}X$ and $\breve{\mathsf{T}}{\restriction}X$ and $\neg C(Z){\restriction}X$, contradicting $(*)$. □

We typically refer to this lemma to justify a claim such as $\mathsf{p}_1^2 \Rightarrow \mathsf{p}_1$: as trivially, $\mathsf{T}^\epsilon \vdash \mathsf{p}_1(\breve{\mathsf{T}}) \to \mathsf{p}_1(\breve{\mathsf{T}})$, and $\mathsf{p}_1(\breve{\mathsf{T}})$ is Π^1_2, the above lemma yields $\mathsf{T}^\epsilon \vdash \mathsf{p}_1^2(\breve{\mathsf{T}}) \to \mathsf{p}_1(\breve{\mathsf{T}})$, hence indeed $\mathsf{p}_1^2 \Rightarrow \mathsf{p}_1$.

Remark I.2.13. *If* $\breve{\mathsf{T}}$ *is* Π^1_{n+2}*, then* $\mathsf{T}^\epsilon \vdash \mathsf{p}_{n+1}(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}$ *by Lemma I.2.12. As by Convention I.2.4,* $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to (\mathsf{ACA})$*, and since further* (ACA) *is* Π^1_2*, we also have that* $\mathsf{T}^\epsilon \vdash \mathsf{p}_1(\breve{\mathsf{T}}) \to (\mathsf{ACA})$*. Moreover, we will prove (Lemma III.6.5 (i)), that* $\mathsf{Op} \Rightarrow \mathsf{p}_1$ *for each operation build from basic operations. Consequently,* $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to (\mathsf{ACA})$ *for all operations* Op *and open sentences* $\breve{\mathsf{T}}$ *we consider. This justifies Convention I.2.4.*

The next Lemma exhibits another key property, which is generalized later.

Lemma I.2.14. *If* $\breve{\mathsf{T}}'$ *is* Π^1_{m+1}*, then* $\mathsf{T}^\epsilon \vdash \mathsf{p}_{m+1}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{p}_{m+1}(\breve{\mathsf{T}} \wedge \breve{\mathsf{T}}')$*.*

Proof By the above remark, $\mathsf{p}_{m+1}(\breve{\mathsf{T}})$ implies arithmetical comprehension. Now let $\breve{\mathsf{T}}'$ be Π^1_{m+1}. We just show the case $m > 0$; the case $m = 0$ is similar but simpler. We assume $\mathsf{p}_{m+1}(\breve{\mathsf{T}})$ and $\breve{\mathsf{T}}'$, and aim for $\mathsf{p}_{m+1}(\breve{\mathsf{T}} \wedge \breve{\mathsf{T}}')$. Assume that $\pi^1_{m+1}(Z, x, e)$. We have to find a set X so that $Z\dot{\in}X$, $\pi^1_{m+1}(Z, x, e){\restriction}X$, $\breve{\mathsf{T}}{\restriction}X$ and $\breve{\mathsf{T}}'{\restriction}X$. As (ACA) is at hand, there is an e' so that $\pi^1_{m+1}(Z, x, e')$ iff $\breve{\mathsf{T}}' \wedge \pi^1_{m+1}(Z, x, e)$. Hence, there is a set X so that $Z\dot{\in}X$, $\pi^1_{m+1}(Z, x, e'){\restriction}X$ and $\breve{\mathsf{T}}{\restriction}X$. Since also $(\mathsf{ACA}){\restriction}X$, and therefore $(\pi^1_{m+1}(Z, x, e') \leftrightarrow \breve{\mathsf{T}}' \wedge \pi^1_{m+1}(Z, x, e)){\restriction}X$, $\pi^1_{m+1}(Z, x, e'){\restriction}X$ implies $\pi^1_{m+1}(Z, x, e){\restriction}X$ and $\breve{\mathsf{T}}'{\restriction}X$, and we are done. □

I.2.2 Representation of operations

We have introduced operations as maps on open L_2-sentences. Hence, with p_1, also the operation $\mathsf{p}_1^2 = \mathsf{p}_1 \circ \mathsf{p}_1$ is explained. But what should p_1^ω be?

In order to answer this question, we first explain how to represent an operation Op by an $\mathsf{L}_2(\mathsf{P})$-formula φ. The idea is that the formula $\breve{\mathsf{T}}$ is mapped to the formula $\varphi\{\breve{\mathsf{T}}{\restriction}U\}$ obtained from φ by replacing each occurrence of $\mathsf{P}(\mathcal{X})$ in φ by the formula $\breve{\mathsf{T}}{\restriction}\mathcal{X}$. Before we give the general definition of when an $\mathsf{L}_2(\mathsf{P})$-formula φ represents an operation, we start by presenting $\mathsf{L}_2(\mathsf{P})$-formulas so that the map $\breve{\mathsf{T}} \mapsto \varphi_{\mathsf{p}_{n+1}}\{\breve{\mathsf{T}}{\restriction}U\}$ coincides with the map p_{n+1}.

Definition I.2.15.

(i) $\varphi_{\mathsf{p}_1} := \forall Z \exists X[Z \dot{\in} X \wedge \mathsf{P}(X)] \wedge \mathsf{pair} \wedge \mathsf{trans}$, *and*

(ii) $\varphi_{\mathsf{p}_{n+2}} := \forall Z \forall x, e \exists X[Z \dot{\in} X \wedge \mathsf{P}(X) \wedge R_{n+2}(X, Z, x, e)] \wedge \mathsf{pair} \wedge \mathsf{trans}$.

Now we address the initial question of what p_1^ω should be. Since p_1^ω is an operation, $\mathsf{p}_1^\omega(\breve{\mathsf{T}})$ is an open L_2-sentence. We emphasize that $\mathsf{p}_1^\omega(\mathsf{T})$ is not the theory $\bigcup_{n\in\mathbb{N}} \mathsf{p}_1^{n+1}(\mathsf{T})$, where $\mathsf{p}_1^1(\mathsf{T}) := \mathsf{p}_1(\mathsf{T})$ and $\mathsf{p}_1^{n+2}(\mathsf{T}) := \mathsf{p}_1(\mathsf{p}_1^{n+1}(\mathsf{T}))$. Instead, p_1^ω is the operation $\breve{\mathsf{T}} \mapsto \varphi_{\mathsf{p}_1^\omega}\{\breve{\mathsf{T}}{\restriction}U\}$, where $\varphi_{\mathsf{p}_1^\omega} := \forall n \vartheta(n)$, and $\vartheta(u)$ is such that for all $\breve{\mathsf{T}}$ (which imply (ACA)),

(i) $\mathsf{T}^\epsilon \vdash \vartheta(1)\{\breve{\mathsf{T}}{\restriction}U\} \leftrightarrow \mathsf{p}_1(\breve{\mathsf{T}})$, and

(ii) $\mathsf{T}^\epsilon \vdash \forall n[\vartheta(n{+}2)\{\breve{\mathsf{T}}{\restriction}U\} \leftrightarrow \mathsf{p}_1(\vartheta(n{+}1)\{\breve{\mathsf{T}}{\restriction}U\})]$.

That an $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ with these properties exists is a consequence of the Representation Theorem I.2.26.

Let us summarize the above discussion.

Definition I.2.16. *The language* $\mathsf{L}_2(\mathsf{P})$ *extends* L_2 *by a fresh relation symbol* $\mathsf{P}(U)$*. Each set variable is a set term, and if* $\mathcal{X}$ *is a set term, then also* $(\mathcal{X})_s$ *for each number term* s*. Each atom of* L_2 *is an atom of* $\mathsf{L}_2(\mathsf{P})$*, and for each set term* $\mathcal{X}$ *of* $\mathsf{L}_2(\mathsf{P})$*,* $\mathsf{P}(\mathcal{X})$ *is an atom of* $\mathsf{L}_2(\mathsf{P})$*. Further,* φ *is an* $\mathsf{L}_2(\mathsf{P}^+)$*-formula, if* φ *contains* P *only positively.*

Since $\mathsf{P}((X)_{\vec{s}})$ is an atom of $\mathsf{L}_2(\mathsf{P})$, the notations $C{\restriction}X$ and $X \models C$ lift canonically to $\mathsf{L}_2(\mathsf{P})$-formulas (cf. Definition I.1.3). For instance, $\forall X \mathsf{P}(X){\restriction}U$ is $\forall x \mathsf{P}((U)_x)$, and $\forall Y \mathsf{P}((Y)_x){\restriction}U$ is $\forall y \mathsf{P}((U)_{y,x})$. Below, we let $\varphi, \psi, \ldots$ range over $\mathsf{L}_2(\mathsf{P})$-formulas.

Definition I.2.17. *Let* φ *and* $\psi(U)$ *be* $\mathsf{L}_2(\mathsf{P})$*-formulas. Then* $\varphi\{\psi(U)\}$ *is the* $\mathsf{L}_2(\mathsf{P})$*-formula obtained from* φ *by replacing each occurrence of* $\mathsf{P}(\mathcal{X})$ *in* φ *by the formula* $\psi(\mathcal{X})$*. We also write* $\varphi_{\psi(U)}$ *for* $\varphi\{\psi(U)\}$*. Further, for each* $\mathsf{L}_2(\mathsf{P})$*-formula* $\vartheta(U)$*,*

$\{\vartheta(U)\}$ takes precedence over quantifiers and logical connective: $\mathcal{Q}y\varphi\{\vartheta(U)\} := \mathcal{Q}y(\varphi\{\vartheta(U)\})$, $\mathcal{Q}Y\varphi\{\vartheta(U)\} := \mathcal{Q}Y(\varphi\{\vartheta(U)\})$, and $(\varphi\, j\, \psi)\{\vartheta(U)\} := \varphi\, j\, \psi\{\vartheta(U)\}$, where j is a connective in $\{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

When we apply this substitution, it is always assumed that $\mathsf{BV}(\varphi) \cap \mathsf{FV}(\psi(U)) = \emptyset$, $\mathsf{FV}(\varphi) \cap \mathsf{FV}(\psi(U)) = \emptyset$, and that if $\mathsf{P}(\mathcal{X})$ occurs in φ, then $\mathsf{FV}(\mathsf{P}(\mathcal{X})) \cap \mathsf{BV}(\psi) = \emptyset$. Observe that if $A(U)$ is an L_2-formula, then $\varphi\{A(U)\}$ is an $\mathsf{L}_2(\mathsf{P})$-formula in which the relation symbol P no longer occurs. Strictly speaking, however, $\varphi\{A(U)\}$ is not an L_2-formula: if e.g. $\mathcal{X}$ is a set term different from a set variable, then $\mathsf{P}(\mathcal{X})\{0 \in U\}$ is the formula $0 \in \mathcal{X}$, which still contains the set term $\mathcal{X}$. Therefore, we identify $\varphi\{A(U)\}$ with the L_2-formula obtained by regarding the set terms $\mathcal{X}$ occurring in $\varphi\{A(U)\}$ as abbreviations according to Definition I.1.2.

Note that in $\varphi\{\psi(U)\}$, the variable U plays the role of a place-holder that indicates where to place the set term $\mathcal{X}$. For all variables U, V, we have that $\varphi\{\psi(U)\} = \varphi\{\psi(V)\}$, and $\mathsf{FV}(\varphi\{\psi(U)\}) = \mathsf{FV}(\varphi) \cup \mathsf{FV}(\psi(U)) \setminus \{U\}$.

Another immediate consequence of the definition of $\varphi\{\psi(U)\}$ is that substitution distributes over quantifiers and logical connectives.

Lemma I.2.18. *For all $\mathsf{L}_2(\mathsf{P})$-formulas φ, ψ and ϑ, $(\mathcal{Q}y\varphi)\{\vartheta(U)\} = \mathcal{Q}y\varphi\{\vartheta(U)\}$, $(\mathcal{Q}Y\varphi)\{\vartheta(U)\} = \mathcal{Q}Y\varphi\{\vartheta(U)\}$, and for each $j \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, $(\varphi\, j\, \psi)\{\vartheta(U)\} = \varphi\{\vartheta(U)\}\, j\, \psi\{\vartheta(U)\}$.*

The next Lemma guarantees that under mild assumptions, for each $\mathsf{L}_2(\mathsf{P}^+)$-formula φ, $\breve{\mathsf{T}} \mapsto \varphi\{\breve{\mathsf{T}}{\restriction}U\}$ is an operation in the sense of Definition I.2.1.

Lemma I.2.19. *If φ is an $\mathsf{L}_2(\mathsf{P}^+)$-formula, and $A(U)$, $B(U)$ are L_2-formulas, then $\mathsf{T}^\epsilon + \mathsf{trans} \vdash \forall X(A(X) \rightarrow B(X)) \rightarrow (\varphi\{A(U)\} \rightarrow \varphi\{B(U)\})$.*

Proof Assume that $\forall X(A(X) \rightarrow B(X))$. We show $\varphi^{\rightarrow} := \varphi\{A(U)\} \rightarrow \varphi\{B(U)\}$ by induction on the build-up of φ, tacitly using the above lemma.

(i) φ does not contain P. Then $\varphi^{\rightarrow} = \varphi$ and the claim holds trivially.

(ii) $\varphi = \mathsf{P}(\mathcal{X})$ for some set term $\mathcal{X}$. Then $\varphi^{\rightarrow}$ is $A(\mathcal{X}) \rightarrow B(\mathcal{X})$, which follows from $\forall X(A(X) \rightarrow B(X))$ as $\mathcal{X}$ is a set by trans.

(iii) $\varphi = \varphi_1\, j\, \varphi_2$ for $j \in \{\wedge, \vee\}$. By I.H., $\varphi_1\{A(U)\} \rightarrow \varphi_1\{B(U)\}$ and $\varphi_2\{A(U)\} \rightarrow \varphi_2\{B(U)\}$. The claim follows as $\varphi\{A(U)\} = \varphi_1\{A(U)\}\, j\, \varphi_2\{A(U)\}$ implies $\varphi_1\{B(U)\}\, j\, \varphi_2\{B(U)\} = \varphi\{B(U)\}$.

(iv) $\varphi = \forall Y\psi(Y)$. By I.H., $\psi(Y)\{A(U)\} \rightarrow \psi(Y)\{B(U)\}$. By logic, we conclude $\forall Y\psi(Y)\{A(U)\} \rightarrow \psi(Y)\{B(U)\}$ and $\forall Y\psi(Y)\{A(U)\} \rightarrow \forall Y\psi(Y)\{B(U)\}$, and $\varphi^{\rightarrow}$ readily follows.

(v) $\varphi = \exists Y \psi(Y)$. By I.H., $\psi(Y)\{A(U)\} \to \psi(Y)\{B(U)\}$, and $\psi(Y)\{A(U)\} \to \exists Y \psi(Y)\{B(U)\}$ and $\exists Y \psi(Y)\{A(U)\} \to \exists Y \psi(Y)\{B(U)\}$, thus $\varphi^{\to}$.

$\square$

If φ is an open $\mathsf{L}_2(\mathsf{P})$-sentence, then $\mathsf{Op}^{\varphi}(\breve{\mathsf{T}}) := \varphi\{\breve{\mathsf{T}}\restriction U\}$, and if φ is introduced as $\varphi(\vec{u})$, then $\mathsf{Op}^{\varphi}_{\vec{t}}(\breve{\mathsf{T}}) := \mathsf{Op}^{\varphi}(\breve{\mathsf{T}})[\vec{t}/\vec{u}]$. The following is an easy consequence of Lemma I.2.19.

Theorem I.2.20. *Assume that $\varphi := \varphi(\vec{u})$ is an open $\mathsf{L}_2(\mathsf{P})$-sentence that contains P, P occurs only positively in φ, and $\mathsf{T}^{\epsilon} \vdash \varphi\{\top\} \to \mathsf{pair} \wedge \mathsf{trans}$. Then Op^{φ} and $\mathsf{Op}^{\varphi}_{\vec{t}}$ are operations in the sense of Definition I.2.1.*

Proof We have $\mathsf{Op}^{\varphi}(\breve{\mathsf{T}}) := \varphi\{\breve{\mathsf{T}}\restriction U\}$. Since $\mathsf{FV}_1(\varphi) = \emptyset$, the set variable in each occurrence of $\mathsf{P}(\mathcal{X})$ in φ is within the scope of a set quantifier. Therefore, this set variable is still bound after substituting $\breve{\mathsf{T}}\restriction\mathcal{X}$ for $\mathsf{P}(\mathcal{X})$. So $\mathsf{Op}^{\varphi}(\breve{\mathsf{T}})$ is an open L_2-sentence. Since it is assumed that the variables occurring in φ and $\breve{\mathsf{T}}\restriction U$ are disjoint, we have $\mathsf{FV}_0(\breve{\mathsf{T}}) \subseteq \mathsf{FV}_0(\mathsf{Op}^{\varphi}(\breve{\mathsf{T}}))$ and $\mathsf{FV}_0(\breve{\mathsf{T}}) \subseteq \mathsf{FV}_0(\mathsf{Op}^{\varphi}_{\vec{t}}(\breve{\mathsf{T}}))$. To show (ii) of Definition I.2.1, let $C(\vec{u}) := \forall X(\breve{\mathsf{T}}\restriction X \to \breve{\mathsf{T}}'\restriction X) \to (\mathsf{Op}^{\varphi}_{\vec{u}}(\breve{\mathsf{T}}) \to \mathsf{Op}^{\varphi}_{\vec{u}}(\breve{\mathsf{T}}'))$. By Lemma I.2.19 ($\breve{\mathsf{T}}\restriction U$ and $\breve{\mathsf{T}}'\restriction U$ take the roles of $A(U)$ and $B(U)$), we have that $\mathsf{T}^{\epsilon} + \mathsf{trans} \vdash C$. Since $\mathsf{T}^{\epsilon} \vdash \forall X[(\breve{\mathsf{T}}\restriction X) \to \top\restriction X]$, also $\mathsf{T}^{\epsilon} \vdash \mathsf{Op}^{\varphi}(\breve{\mathsf{T}}) \to \mathsf{Op}^{\varphi}\{\top\}$. As $\mathsf{Op}^{\varphi}\{\top\}$ is $\varphi\{\top\}$, $\mathsf{T}^{\epsilon} \vdash \mathsf{Op}^{\varphi}(\breve{\mathsf{T}}) \to \mathsf{trans}$ is by assumption. Thus also $\mathsf{T}^{\epsilon} \vdash C(\vec{u})$. Hence Op^{φ} and $\mathsf{Op}^{\varphi}_{\vec{t}}$ are operations. $\square$

Now, we say what we mean by "φ represents the operation Op^{φ}".

Definition I.2.21. *We say that φ represents the operation $\mathsf{Op}^{\varphi}(\breve{\mathsf{T}}) := \varphi\{\breve{\mathsf{T}}\restriction U\}$, if φ is an open $\mathsf{L}_2(\mathsf{P})$-sentence that contains P, P occurs only positively in φ, and $\mathsf{T}^{\epsilon} \vdash \varphi\{\top\} \to \mathsf{pair} \wedge \mathsf{trans}$.*

Lemma I.2.22. *Assume that φ, $\psi(U)$ are $\mathsf{L}_2(\mathsf{P})$-formulas, $A(U), B(U)$ L_2-formulas, and $A(U)$ is arithmetical. Then,*

(i) $(\varphi\restriction\mathcal{X})\{A(U)\} = \varphi_{A(U)}\restriction\mathcal{X}$,

(ii) $(\varphi_{\psi(V)})\{B(U)\} = \varphi\{\psi(V)_{B(U)}\}$.

Proof Both claims are shown by induction on the build-up of φ. We just show some cases, starting with the first claim. Thus we assume that $A(U)$ is an arithmetical.

(i) If P does not occur in φ, then $(\varphi\restriction\mathcal{X})\{A(U)\} = \varphi\restriction\mathcal{X} = \varphi_{A(U)}\restriction\mathcal{X}$.

(ii) $\varphi = [\neg]\mathsf{P}(\mathcal{Y})$ for some set term $\mathcal{Y}$. Then $(\varphi\restriction\mathcal{X})\{A(U)\} = [\neg]\mathsf{P}(\mathcal{Y})\{A(U)\} = [\neg]A(\mathcal{Y})$, and also $\varphi_{A(U)}\restriction\mathcal{X} = [\neg]A(\mathcal{Y})\restriction\mathcal{X} = [\neg]A(\mathcal{Y})$ as A is arithmetical.

(iii) $\varphi = \mathcal{Q}Y\psi(Y)$. Unwinding definitions and using Lemma I.2.18, we see that $\varphi{\restriction}X = (\mathcal{Q}Y\psi(Y)){\restriction}\mathcal{X} = (\mathcal{Q}Y\dot{\in}\mathcal{X})(\psi(Y){\restriction}\mathcal{X}) = \mathcal{Q}y\psi((\mathcal{X})_y){\restriction}\mathcal{X}$, and therefore $(\varphi{\restriction}\mathcal{X})\{A(U)\} = (\mathcal{Q}y\psi((\mathcal{X})_y){\restriction}\mathcal{X})\{A(U)\} = \mathcal{Q}y\psi((\mathcal{X})_y){\restriction}\mathcal{X}\{A(U)\}$. Further, $(\varphi_{A(U)}){\restriction}\mathcal{X} = ((\mathcal{Q}Y\psi(Y))_{A(U)}){\restriction}\mathcal{X} = (\mathcal{Q}Y\psi(Y)_{A(U)}){\restriction}\mathcal{X} = \mathcal{Q}y\psi((\mathcal{X})_y)_{A(U)}{\restriction}\mathcal{X}$. By I.H. we have

$$\psi((\mathcal{X})_y){\restriction}\mathcal{X}\{A(U)\} = \psi((\mathcal{X})_y)_{A(U)}{\restriction}\mathcal{X}.$$

Quantifying y on both sides yields the claim.

Now for the second claim. If P does not occur in φ, then again, the claim is readily checked. For the induction step, we exemplarily consider the case $\varphi = \mathcal{Q}Y\vartheta(Y)$. Then, $(\mathcal{Q}Y\vartheta(Y))_{\psi(V)}\{B(U)\} = (\mathcal{Q}Y\vartheta(Y)_{\psi(V)})\{B(U)\} = \mathcal{Q}Y\vartheta(Y)_{\psi(V)}\{B(U)\} =_{I.H.} \mathcal{Q}Y\vartheta(Y)\{\psi(V)_{B(U)}\} = (\mathcal{Q}Y\vartheta(Y))\{\psi(V)_{B(U)}\} = \varphi\{\psi(V)_{B(U)}\}$. □

Example I.2.23. *Assume that φ and ψ represent the operations Op^{φ} and Op^{ψ}, respectively. Then, as a direct consequence of Lemma I.2.22, $\varphi \circ \psi := \varphi\{\psi{\restriction}U\}$ represents $\mathsf{Op}^{\varphi} \circ \mathsf{Op}^{\psi}$. We namely have that $(\varphi \circ \psi)\{\breve{\mathsf{T}}{\restriction}U\} = \varphi_{\psi{\restriction}V}\{\breve{\mathsf{T}}{\restriction}U\} =_{(ii)} \varphi\{(\psi{\restriction}V)_{\breve{\mathsf{T}}{\restriction}U}\} =_{(i)} \varphi\{\psi_{\breve{\mathsf{T}}{\restriction}U}{\restriction}V\} = \varphi\{\mathsf{Op}^{\psi}(\breve{\mathsf{T}}){\restriction}V\} = (\mathsf{Op}^{\varphi} \circ \mathsf{Op}^{\psi})(\breve{\mathsf{T}})$.*

The reminder of this section is devoted to the formulation of the Representation Theorem. It claims that we can construct from an $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ representing operations Op_u, an $\mathsf{L}_2(\mathsf{P})$-formula $\varphi(u)$ which represents new operations Op'_u, that are obtained by transfinite compositions of operations Op_u. Its proof is rather technical and was thus moved to the appendix.

The idea is the following. Assume that $(Q, \prec)$ is a primitive recursive well-founded ordering with least element 0, that is, $\mathsf{field}(\prec) = Q$, and for each $q \in Q$, $0 \preceq q$ and $\mathsf{Wo}_{\prec}(q)$. By recursion on $\prec$, we want to define for each $q \in Q^* := Q \setminus \{0\}$ an operation Op_q in terms of initial operations $\mathsf{Op}^{\vartheta}_u$ and some of the previously defined operations $\{\mathsf{Op}_p : p \prec q, 0 \neq p\}$. More specifically, given an $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ that represents initial operations $\mathsf{Op}^{\vartheta}_u$, a primitive recursive subset $\rightsquigarrow$ of $\prec$ and a primitive recursive function $f(u, v)$, then we want an $\mathsf{L}_2(\mathsf{P})$-formula $\varphi(u)$ so that

$$(*) \qquad \mathsf{T}^{\epsilon} \vdash \mathsf{Op}^{\varphi}_u(\breve{\mathsf{T}}) \leftrightarrow 0 \prec u \wedge (\forall v \rightsquigarrow u)[\mathsf{Op}^{\vartheta}_{f(v,u)}(\widehat{\mathsf{Op}}^{\varphi}_v(\breve{\mathsf{T}}))],$$

where $\widehat{\mathsf{Op}}^{\vartheta}_u(\breve{\mathsf{T}}) := (0 = u \wedge \breve{\mathsf{T}}) \vee (0 \neq u \wedge \mathsf{Op}^{\vartheta}_u(\breve{\mathsf{T}}))$, an abbreviation that we use since we cannot represent directly the identity operation by an $\mathsf{L}_2(\mathsf{P})$-formula.

We prove $(*)$ only for relations $\rightsquigarrow$ and $\prec$ where $\prec$ is the transitive closure of $\rightsquigarrow$, and we let $\mathsf{good}(\rightsquigarrow, \prec)$ be an arithmetical L_2-sentence so that $\forall u(0 \prec u \wedge \mathsf{Wo}_{\prec}(u))$ together with $\mathsf{good}(\rightsquigarrow, \prec)$ implies that $\prec$ is the transitive closure of $\rightsquigarrow$.

Definition I.2.24. *Let* $\mathsf{good}(\rightsquigarrow, \prec)$ *be the conjunctions of the following formulas.*

(i) $\forall xy[x \rightsquigarrow y \to x \prec y]$,

(ii) $\forall xyz[x \prec y \wedge y \rightsquigarrow z \to x \prec z]$,

(iii) $(\forall x \prec z)(\exists y \rightsquigarrow z)(x \preceq y)$.

So under the assumption $\forall u(0 \prec u \wedge \mathsf{Wo}_\prec(u))$, (i) and (ii) imply that $\prec$ contains the transitive closure of $\rightsquigarrow$, and (iii) yields the other direction, namely that if $x \prec z$, there is $\rightsquigarrow$-path from x to z, i.e. $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \ldots \rightsquigarrow x_n = x$ for some $x_0, \ldots, x_n \in \mathsf{field}(\rightsquigarrow)$. Otherwise, there is a $\prec$-minimal element z so that there is an x with $x \prec z$ but there is no $\rightsquigarrow$-path from x to z. By (iii) however, there is a y with $y \rightsquigarrow z$ with $x \preceq y$. Clearly, $x \neq y$, thus $x \prec y \rightsquigarrow z$. By the minimality of z, there is a $\rightsquigarrow$-path from x to y, hence also from x to z!

The following technical notion is convenient to have at hand for the proof of the Representation Theorem.

Definition I.2.25. *An* $\mathsf{L}_2(\mathsf{P})$*-formula* ϑ *strongly implies* p_1*, if* ϑ *represents an operation, and for each* Σ^1_1*-formula* $A(U)$ *of* L_2*,* $\mathsf{T}^\epsilon \vdash \vartheta\{A(U)\} \to \varphi_{\mathsf{p}_1}\{A(U)\}$.

If ϑ strongly implies p_1, then in particular, $\mathsf{Op}^\vartheta \Rightarrow \mathsf{p}_1$, since $\mathsf{Op}^\vartheta(\breve{\mathsf{T}}) = \vartheta\{\breve{\mathsf{T}}{\restriction}U\}$ and $\{\breve{\mathsf{T}}{\restriction}U\}$ is arithmetical and thus also Σ^1_1.

Now the stage is set the state the theorem. Since we find it convenient to have that $\mathsf{Op}^\varphi_u(\breve{\mathsf{T}})$ implies $0 \rightsquigarrow u$ and $\mathsf{Wo}_\prec(u)$, we will defined the formula φ so that it directly implies these properties. Hence, if e.g. $u \notin \mathsf{field}(\prec)$ and thus $\neg(0 \prec u)$, then $\mathsf{Op}^\varphi_u(\breve{\mathsf{T}})$ is inconsistent and therefore proves everything.

Theorem I.2.26 (Representation Theorem)**.** *Let* $\vartheta(u)$ *an open* $\mathcal{L}(\mathsf{P})$*-sentence that strongly implies* p_1*, and* $\prec, \rightsquigarrow, f(v,u)$ *primitive recursive. Then there is an open* $\mathcal{L}(\mathsf{P}^+)$*-sentence* $\varphi(u) := \varphi^{f,\prec,\rightsquigarrow,\vartheta}(u)$ *that strongly implies* p_1*, so that for each* $\breve{\mathsf{T}}$ *(that implies* (ACA)*),* T^ϵ *proves*

(i) $\mathsf{Op}^\varphi_u(\breve{\mathsf{T}}) \to 0 \prec u \wedge \mathsf{Wo}_\prec(u) \wedge \mathsf{good}(\rightsquigarrow, \prec)$,

(ii) $0 \prec u \wedge \mathsf{Wo}_\prec(u) \wedge \mathsf{good}(\rightsquigarrow, \prec) \to [\, \mathsf{Op}^\varphi_u(\breve{\mathsf{T}}) \leftrightarrow (\forall v \rightsquigarrow u)(\mathsf{Op}^\vartheta_{f(v,u)}(\widehat{\mathsf{Op}}^\varphi_v(\breve{\mathsf{T}})))\,]$.

Example I.2.27. *Let* $\vartheta = \varphi_{\mathsf{p}_1}$*,* $\prec$ *some primitive recursive well-ordering and* $\alpha \rightsquigarrow \beta$ *iff* $\beta = \alpha{+}1$ *or* β *is a limit* $\prec$*-bigger than* α*.* f *is irrelevant, as* φ_{p_1} *has no free variables. Then, for* $\varphi := \varphi^{f,\prec,\rightsquigarrow,\vartheta}$*,* $\mathsf{Op}^\varphi_1 = \mathsf{p}_1$*, if* $0 \prec \alpha$*, then* $\mathsf{Op}^\varphi_{\alpha+1} \Leftrightarrow \mathsf{p}_1(\mathsf{Op}^\varphi_\alpha)$*, and if* γ *is a limit, then* $\mathsf{Op}^\varphi_\gamma \Leftrightarrow (\forall \alpha \prec \gamma)\mathsf{p}_1(\widehat{\mathsf{Op}}^\varphi_{1+\alpha})$.

Slightly stretching Definition I.2.2, we wrote $\mathsf{Op}^\varphi_\gamma \Leftrightarrow (\forall \alpha \prec \gamma)\mathsf{p}_1(\widehat{\mathsf{Op}}^\varphi_{1+\alpha})$ to indicate that for each $\breve{\mathsf{T}}$ (that implies (ACA)), $\mathsf{T}^\epsilon \vdash \mathsf{Op}^\varphi_\gamma(\breve{\mathsf{T}}) \leftrightarrow (\forall \alpha \prec \gamma)\mathsf{p}_1(\widehat{\mathsf{Op}}^\varphi_{1+\alpha}(\breve{\mathsf{T}}))$.

Corollary I.2.28. *If ϑ is an open $\mathsf{L}_2(\mathsf{P})$-sentence that strongly implies p_1, then there is an open $\mathsf{L}_2(\mathsf{P})$-sentence $\varphi(u)$ which strongly implies p_1, so that $\mathsf{Op}_1^{\varphi} \Leftrightarrow \mathsf{Op}^{\vartheta}$, if $0 \prec \alpha$, then $\mathsf{Op}_{\alpha+1}^{\varphi} \Leftrightarrow \mathsf{Op}^{\vartheta}(\mathsf{Op}_{\alpha}^{\varphi})$, and if γ is a limit, $\mathsf{Op}_{\gamma}^{\varphi} \Leftrightarrow (\forall \alpha \prec \gamma)\mathsf{Op}^{\vartheta}(\mathsf{Op}_{1+\alpha}^{\varphi})$.*

This justifies the following notation: if Op is represented by ϑ, then we write $\mathsf{Op}^{\alpha}(\breve{\mathsf{T}})$ for $\mathsf{Op}_{\alpha}^{\varphi}(\breve{\mathsf{T}})$, where $\varphi(u)$ is the formula provided by the above corollary. Note also that then $\mathsf{Op}^{\gamma} \Leftrightarrow (\forall \alpha < \gamma)\mathsf{Op}^{\alpha+1}$.

I.3 The functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$

In this section, we introduce the family $(\mathsf{It}_{n+1} : n \in \mathbb{N})$ of functionals. We assume that the reader is familiar with ordinals, normal functions and closed unbounded sets. The used properties of these concepts are covered e.g. in Pohlers [11], Chapter I, Section 6.

Below, Ω refers to the first uncountable ordinal. To simplify the notation, we identify a set $O \subseteq \Omega$ with the strictly monotone function f_O enumerating the elements of O, and conversely, a strictly monotone function $f : \Omega \to \Omega$ is identified with its range $\mathsf{rng}(f)$. Hence, if $f, g : \Omega \to \Omega$ are strictly monotone, then $f \subseteq g$ is short for $\mathsf{rng}(f) \subseteq \mathsf{rng}(g)$. Further, we tacitly use that f is normal iff $\mathsf{rng}(f)$ is closed unbounded, and that $\bigcap_{i \in \mathbb{N}} f_i$ is normal if each $f_i : \Omega \to \Omega$ is normal.

Henceforth, we regard the ordinals in Ω also as type-0 functionals, and the functions $f : \Omega \to \Omega$ as type-1 functionals. By ${}^{X}Y$ we denote the set of functions from X to Y. For $X \subseteq {}^{\Omega}\Omega$, $F \in {}^{X}X$ is a type-2 functional. And if X is a set of type-n functionals, then $F \in {}^{X}X$ is a type-n+1 functional.

Definition I.3.1. *Assume that $F_0, \ldots, F_n$ are functionals (or functions or ordinals). We let $(F_0) := F_0$, if $F_1 \in \mathsf{dom}(F_0)$, then $(F_0, F_1) := F_0(F_1)$, and further, for $0 < i < n$, if $F_{i+1} \in \mathsf{dom}(F_0, F_1, \ldots, F_i) := \mathsf{dom}((F_0, F_1, \ldots, F_i))$, then $(F_0, F_1, \ldots, F_i, F_{i+1}) := (F_0(F_1), F_2, \ldots, F_{i+1})$. We also write $F_0[F_1, \ldots, F_{n+1}]$ for $(F_0, F_1, \ldots, F_{n+1})$.*

For instance, if the function $f : \Omega \to \Omega$ is in the domain of the functional F, then $(F, f, \alpha) = F[f, \alpha] = (F(f))(\alpha)$. If $F_1, \ldots, F_n$ are suitable functionals, then $(F_1, \ldots, F_n)$ always denotes the functional defined above, and never an n-tuple.

All functionals of type-n+2 that are relevant for our purpose are build from functionals that do just one simple thing: they iterate functions and functionals, respectively.

Definition I.3.2 (Iteration)**.** *Below, γ ranges over $\mathsf{lim}(\Omega)$, the limit ordinals in Ω. Further, $\mathsf{id}_X := \{(x, x) : x \in X\}$.*

(i) *For* $h : \Omega \to \Omega$, $h^0 := \mathsf{id}_\Omega$, $h^{\alpha+1} := h \circ h^\alpha$ *and* $h^\gamma(\beta) := \mathsf{sup}_{\alpha<\gamma} h^\alpha(\beta)$.

(ii) *For a type-n+1 functional* F, $F^0 := \mathsf{id}_{\mathsf{dom}(F)}$, $F^{\alpha+1} := F \circ F^\alpha$, *and whenever* $f \in \mathsf{dom}(F_1), F_1 \in \mathsf{dom}(F_2), \ldots, F_n \in \mathsf{dom}(F)$,

$$(F^\gamma, F_n, \ldots, F_1, f) := \bigcap_{\alpha<\gamma} (F^\alpha, F_n, \ldots, F_1, f).$$

Furthermore, we consider only normal functions f with $\mathsf{rng}(f) \subseteq \mathsf{lim}(\Omega)$, and we want that on such functions, our functionals thin out a function in the following way: $F(f) \subseteq f$ and $f(0) \notin F[f, 0]$; hence $F[f, 0] > f(0)$. We call such an F *strictly inclusive*. Below, we lift this notion to higher types.

Definition I.3.3. *By recursion on* n *we define the sets* $\Omega^{(n)}$ *and explain when a functional* $F : \Omega^{(n)} \to \Omega^{(n)}$ *is called strictly inclusive.*

(i) $\Omega^{(0)} := \{f : \Omega \to \Omega \mid f \text{ normal }, f \subseteq \mathsf{lim}(\Omega)\}$.

(ii) $F : \Omega^{(0)} \to \Omega^{(0)}$ *is strictly inclusive, if* $(\forall f \in \Omega^{(0)})(f(0) \notin F(f) \subseteq f)$.

(iii) $\Omega^{(n+1)} := \{F \in {}^{\Omega^{(n)}}\Omega^{(n)} : F \text{ is strictly inclusive } \}$.

(iv) $[F_0, \ldots, F_n] \in \Omega^{(\leq n)} :\Leftrightarrow F_0 \in \Omega^{(n)}, \ldots, F_n \in \Omega^{(0)}$.

(v) $F \in {}^{\Omega^{(n+1)}}\Omega^{(n+1)}$ *is strictly inclusive, if for all* $[\vec{H}, h] \in \Omega^{(\leq n)}$,

$$(\vec{H}, h, 0) \notin (F, \vec{H}, h) \subseteq (\vec{H}, h).$$

It is readily observed that for each n, $\Omega^{(n)}$ is closed under composition.

Lemma I.3.4. *For each* n, $\Omega^{(n)}$ *is closed under composition.*

Proof This is trivial for $n = 0$. If $F, G \in \Omega^{(n+1)}$, then clearly $F \circ G : \Omega^{(n)} \to \Omega^{(n)}$. It remains to check that $F \circ G$ is strictly inclusive. Let $[H, \vec{H}, h] \in \Omega^{(\leq n)}$. Then, $(F \circ G, H, \vec{H}, h) = (F, G(H), \vec{H}, h) \subseteq (G(H), \vec{H}, h) = (G, H, \vec{H}, h) \subseteq (H, \vec{H}, h)$, and $(H, \vec{H}, h, 0) \notin (G, H, \vec{H}, h) \supseteq (F \circ G, H, \vec{H}, h)$. □

Convention I.3.5. *Unless stated otherwise,* f, g *range over* $\Omega^{(0)}$, n *ranges over finite ordinals,* $\alpha, \beta, \ldots$ *range over ordinals in* Ω, *and* γ, γ' *range over* $\mathsf{lim}(\Omega)$. *Further,* $f \leq g := \forall\alpha[f(\alpha) \leq g(\alpha)]$, *and for* $F, G \in \Omega^{(n+1)}$, $F \subseteq G$ *iff for all* $[\vec{H}, h] \in \Omega^{(\leq n)}$, $(F, \vec{H}, h) \subseteq (G, \vec{H}, h)$, *and* $F \leq G$ *iff for all* $[\vec{H}, h] \in \Omega^{(\leq n)}$, $(F, \vec{H}, h) \leq (G, \vec{H}, h)$.

Iterating a normal function f does not lead very far as $f^\omega = f' := \{\alpha : f(\alpha) = \alpha\}$ (cf. Lemma I.3.15), and hence $f^{\omega+1} = f^\omega$. Therefore, we iterate the *fixed point free companion* f_- of f instead.

Definition I.3.6 (Fixed point free companion)**.** *For a function $h : \Omega \to \Omega$, we denote by* $\mathsf{fix}(h) := h' := \{\alpha : h(\alpha) = \alpha\}$ *the set of fixed points of h, and $h_- := h \setminus h'$ is the fixed point free companion of h (so h_- is the function that enumerates the set* $\mathsf{rng}(h) \setminus \{\alpha : h(\alpha) = \alpha\}$*).*

We start with a simple observation regarding fixed points, and then state how f relates to its fixed point free companion.

Lemma I.3.7. *If $\alpha \notin f'$, then $f(\alpha) \notin f'$. If $f \in \Omega^{(0)}$, then $f(\alpha){+}1 < f(\alpha{+}1)$, $f(\alpha{+}1) \notin f'$ and $f(\alpha{+}1) \in f_-$.*

Proof If $\alpha \notin f'$, then $\alpha < f(\alpha)$, so $f(\alpha) < f(f(\alpha))$, thus $f(\alpha) \notin f'$. If $f \in \Omega^{(0)}$, then $f(\alpha){+}1 < f(\alpha{+}1)$. Since $\alpha{+}1 \notin f' \subseteq f$, we have $f(\alpha{+}1) \notin f'$ and $f(\alpha{+}1) \in f_-$. $\square$

Remark I.3.8. *As $f \in \Omega^{(0)}$ entails $\omega\alpha \leq f(\alpha)$, each fixed point γ of f satisfies $\omega\gamma \leq f(\gamma) = \gamma \leq \omega\gamma$, and is therefore of the form $\gamma = \omega^\omega\beta$ for some $\beta > 0$.*

Next, we see that f and f_- differ only on ordinals α of the form $\gamma{+}n$ for some $\gamma \in f'$.

Lemma I.3.9. *For each $f \in \Omega^{(0)}$,*

$$f_-(\alpha) = \begin{cases} f(\alpha) & : \alpha < \omega \vee \alpha = \gamma{+}n \text{ for some } \gamma \notin f' \text{ and some } n < \omega, \\ f(\alpha{+}1) & : \alpha = \gamma{+}n \text{ for some } \gamma \in f' \text{ and some } n < \omega. \end{cases}$$

Proof The claim is shown by transfinite induction on α. As $f(0) \notin f'$, $f(0) = f_-(0)$. Next, we show the limit case. Fix some γ and assume that the claim holds for each $\alpha < \gamma$. First observe, that then $\{f_-(\beta) : \beta < \gamma\} = \{f(\beta) : \beta < \gamma\} \cap f_-$, and since for $\beta < \gamma$, $f(\beta) \leq f_-(\beta) \leq f(\beta{+}1)$, also $\mathsf{sup}(\{f_-(\beta) : \beta < \gamma\}) = \mathsf{sup}(\{f(\beta) : \beta < \gamma\})$. We show that the claim holds for γ by distincting the cases $\gamma \notin f'$ and $\gamma \in f'$.

(i) $\gamma \in f'$. Then $f(\gamma) \notin f_-$, and $\gamma < f_-(\gamma) = \mathsf{min}(f_- \setminus \{f_-(\beta) : \beta < \gamma\}) =_{I.H.} \mathsf{min}(f_- \setminus \{f(\beta) : \beta < \gamma\}) = \mathsf{min}(f \setminus (\{f(\beta) : \beta \leq \gamma\} \cup f')) = f(\gamma{+}1)$, since $f(\gamma{+}1) \notin f'$.

(ii) $\gamma \notin f'$. Then $f(\gamma) \in f_-$. As $f \leq f_-$, there is some $\delta \leq \gamma$ so that $f_-(\delta) = f(\gamma) = \mathsf{sup}\{f(\beta) : \beta < \gamma\} =_{I.H.} \mathsf{sup}\{f_-(\beta) : \beta < \gamma\}$. Thus $\delta = \gamma$, and $f_-(\gamma) = f(\gamma)$.

The successor case causes no problems: if for some $m \in \{0, 1\}$, $f_-(\alpha) = f(\alpha{+}m)$, then, as $f(\alpha{+}m{+}1) \in f_-$, also $f_-(\alpha{+}1) = f(\alpha{+}m{+}1)$. $\square$

Now the stage is set to define the functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$. Further, we introduce an auxiliary functional sh which shifts the domain of a function from Ω to $\mathsf{lim}(\Omega)$, and nicely relates the functionals it and fix.

Definition I.3.10 (The type-n+2 functionals It_{n+1}).

(i) $\mathsf{sh}[f,\alpha] := f(\omega(1+\alpha))$.

(ii) $\mathsf{it}[f,\alpha] := (f_-)^{2+\alpha}(0)$, *and* $\mathsf{It}_1 := \mathsf{it}$.

(iii) $\mathsf{It}_{n+2}[F_0,\dots,F_n,f,\alpha] := (F_0^{2+\alpha},F_1,\dots,F_n,f,0)$, *for all* $[\vec{F},f] \in \Omega^{(\leq n+1)}$, *and* $\mathsf{It} := \mathsf{It}_2$.

We defined $\mathsf{it}[f,0] := (f_-)^2(0) = f^2(0)$ to have $f(0) < \mathsf{it}[f,0]$ (i.e. $\mathsf{it}[f,0] \notin f$). For the same reason, we set $\mathsf{It}_{n+2}[F,\vec{F},f,\alpha] := (F^{2+\alpha},\vec{F},f,0)$. Had we defined $\mathsf{It}_{n+2}[F,\vec{F},f,\alpha]$ to be $(F^{1+\alpha},\vec{F},f,0)$, then $\mathsf{It}_{n+2}[F,\vec{F},f,0] = (F,\vec{F},f,0)$, and It_{n+2} would not be strictly inclusive.

Next, we show that sh and it are in $\Omega^{(1)}$, and that $\mathsf{It}_{n+2} \in \Omega^{(n+2)}$.

Lemma I.3.11. $\mathsf{sh} \in \Omega^{(1)}$ *and* $\mathsf{it} \in \Omega^{(1)}$.

Proof The first claim is obvious, so we just show that $\mathsf{it} \in \Omega^{(1)}$. Assume that $f \in \Omega^{(0)}$. Firstly, we show that $\mathsf{it}(f)$ is strictly inclusive. We have already discussed that $\mathsf{it}(f,0) \notin f$. That $\mathsf{it}(f) \subseteq f$ is readily seen by induction on α. If α is zero or a successor, this follows from $f_- \subseteq f$ and the definition of it, and if α is a limit and for each $\beta < \alpha$, $\mathsf{it}(f,\beta) \in f$, then $\mathsf{it}(f,\gamma) \in f$ follows as $\mathsf{rng}(f)$ is closed. Secondly, we show that $\mathsf{it}(f) \in \Omega^{(0)}$. $\mathsf{it}(f)$ is continuous by definition, and $\mathsf{it}(f)$ is strictly monotone since $\alpha < f_-(\alpha)$ ($\alpha \leq f(\alpha) \leq f_(\alpha)$ and $\alpha = f^-(\alpha)$ is impossible). □

Now, we show that also $\mathsf{It}_{n+2} \in \Omega^{(n+2)}$.

Lemma I.3.12. *If* $F \in \Omega^{(n+1)}$, *then* $F^{1+\alpha} \in \Omega^{(n+1)}$. *Further, for all* $[\vec{F},f] \in \Omega^{(\leq n)}$ *and* $d_{F,\vec{F},f}(\alpha) := (F^\alpha,\vec{F},f,0)$, *we have* $d_{F,\vec{F},f} \in \Omega^{(0)}$.

Proof We just consider the case $n = 0$, as the proof works exactly the same for $n > 0$. The first claim is shown by transfinite induction. The case $\alpha = 0$ is trivial, and the successor case follows since $\Omega^{(n+1)}$ is closed under composition. For the limit case, observe that if for each $\alpha < \gamma$, $f(0) \notin F^\alpha(f) \subseteq f$, then also $f(0) \notin \bigcap_{\alpha<\gamma} F^\alpha(f) = F^\gamma(f) \subseteq f$. And because $F^\alpha(f)$ is closed unbounded for each $\alpha < \gamma$, so is $F^\gamma(f)$. Further, as $F^\alpha[f,0] \notin F^{\alpha+1}(f)$, it follows that $d := d_{F,f}$ is strictly monotone. It remains to show that d is continuous: for $\alpha < \gamma$, $d(\alpha) \leq (\bigcap_{\alpha\leq\xi<\gamma} F^\xi(f))(0) = d(\gamma)$, so $\delta := \mathsf{sup}_{\alpha<\gamma} d(\alpha) \leq d(\gamma)$. On the other hand, $F^\beta(f)$ is club for each $\beta < \gamma$. Therefore, $\delta = \mathsf{sup}_{\beta\leq\alpha<\gamma} d(\alpha) \in F^\beta(f)$ for each $\beta < \gamma$, i.e. $\delta \in F^\gamma(f)$, and so $\delta \geq F^\gamma[f,0]$. □

Corollary I.3.13. $\mathsf{It}_{n+1} \in \Omega^{(n+1)}$.

Proof The case $n = 0$ is due to Lemma I.3.11. As $\mathsf{It}_{n+2}[F, \vec{F}, f, \alpha] = d_{F,\vec{F},f}(2+\alpha)$, $\mathsf{It}_{n+2} \in {}^{\Omega^{(n+1)}}\Omega^{(n+1)}$. As with $F \in \Omega^{(n)}$ also $F^{1+\alpha} \in \Omega^{(n+1)}$, $\mathsf{It}_{n+2}(F, \vec{F}, f, \alpha) = (F^{2+\alpha}, \vec{F}, f, 0) \in (F^{2+\alpha}, \vec{F}, f) \subseteq (F, \vec{F}, f)$. Since $\mathsf{It}_{n+2}[F, \vec{F}, f, 0] = (F^2, \vec{F}, f, 0) \notin (F, \vec{F}, f)$, It_{n+2} is also strictly inclusive. $\square$

Lemma I.3.14. *Let $F \in \Omega^{(n+1)}$. Then, $F^\alpha \circ F^\beta = F^{\beta+\alpha}$, in particular $F^\gamma \circ F = F^\gamma$.*

Proof For each β, we show the claim by induction on α: for $\alpha = 0$ there is nothing to show, $F^{\alpha+1} \circ F^\beta =_{I.H.} F \circ F^{\beta+\alpha} = F^{\beta+\alpha+1}$, and for $[\vec{F}, f] \in \Omega^{(\leq n)}$, $(F^\gamma \circ F^\beta, \vec{F}, f) = \bigcap_{\xi<\gamma}(F^\xi \circ F^\beta, \vec{F}, f) =_{I.H.} \bigcap_{\xi<\gamma}(F^{\beta+\xi}, \vec{F}, f) = (F^{\beta+\gamma}, \vec{F}, f)$. $\square$

Next, we relate the functionals it and sh to fix and prove some simple properties. Among other things, the next lemma tells us that $(\mathsf{sh} \circ \mathsf{it}) = \mathsf{fix}$: consequently, $\mathsf{it}[g, \gamma] = g(\mathsf{it}[g, \gamma])$, and if $\alpha \notin \mathsf{lim}(\Omega)$, $\mathsf{it}[g, \alpha] \notin g'$ and so $\mathsf{it}[g, \alpha{+}1] = g(\mathsf{it}[g, \alpha])$.

Lemma I.3.15. *(i) $(\mathsf{fix} \circ \mathsf{sh}) = \mathsf{fix}$, (ii) $(\mathsf{sh} \circ \mathsf{it}) = \mathsf{fix}$ and (iii) $\mathsf{it} \subseteq \mathsf{sh}$.*

Proof (i) $\gamma \in (\mathsf{fix} \circ \mathsf{sh})(f)$ iff $\gamma = \omega\gamma \wedge f(\gamma) = \gamma$ iff $\gamma \in f'$ (cf. Remark I.3.8).
(ii) Note that $\beta{+}1 < f_-(\beta)$, and thus also $f_-^{\alpha+1}(0){+}1 < f_-^{\alpha+2}(0)$. Using Lemma I.3.9, we conclude that $f(f_-^{\alpha+1}(0)) \leq f_-(f_-^{\alpha+1}(0)) \leq f(f_-^{\alpha+2}(0))$. Hence, $\mathsf{it}(f, \gamma) = \mathsf{sup}_{\alpha<\gamma} f_-^{\alpha+2}(0) = \mathsf{sup}_{\alpha<\gamma} f_-(f_-^{\alpha+1}(0)) = \mathsf{sup}_{\alpha<\gamma} f(f_-^{\alpha+1}(0))$, and since f is normal, $\mathsf{sup}_{\alpha<\gamma} f(f_-^{\alpha+1}(0)) = f(\mathsf{sup}_{\alpha<\gamma} f_-^{\alpha+1}(0)) = f(\mathsf{it}[f, \gamma])$. To see that each $\gamma' \in f'$ is of the form $\mathsf{it}[f, \gamma]$, we show that no fixed point of f is strictly between $\mathsf{it}[f, \gamma]$ and $\mathsf{it}[f, \gamma{+}\omega]$. Thereto, we let $\gamma_0 := \mathsf{it}[f, \gamma]$, and show that by induction on n that $f^{n+1}(\gamma_0{+}1) = \mathsf{it}[f, \gamma{+}n{+}1] \notin f'$. For $n = 0$, $f(\gamma_0{+}1) \notin f'$ and $f(\gamma_0{+}1) = f_-(\gamma_0) = f_-(\mathsf{it}[f, \gamma]) = \mathsf{it}[f, \gamma{+}1]$. Next we address the induction step. Since by I.H., $f^{n+1}(\gamma_0{+}1) \notin f'$, also $f^{n+2}(\gamma_0{+}1) \notin f'$. Using Lemma I.3.9 yields $f(f^{n+1}(\gamma_0{+}1)) = f_-(f^{n+1}(\gamma_0{+}1)) =_{IH} f_-(\mathsf{it}([f, \gamma{+}n{+}1]) = \mathsf{it}[f, \gamma{+}n{+}2]$. (iii) If α is not a limit, then the definition of it and Lemma I.3.9 yield that there is a $\beta > 0$ with $\mathsf{it}[f, \alpha] = f(\omega(1{+}\beta)) = \mathsf{sh}[f, \beta]$. And if α is a limit, then $\alpha = \omega(\beta{+}1)$ for some β, therefore $\mathsf{it}[f, \alpha] = (\mathsf{sh} \circ \mathsf{it})[f, \beta] =_{(ii)} \mathsf{fix}[f, \beta] =_{(i)} (\mathsf{fix} \circ \mathsf{sh})[f, \beta]$. Hence, $\mathsf{it}[f, \alpha] \in (\mathsf{fix} \circ \mathsf{sh})(f) \subseteq \mathsf{sh}(f)$. $\square$

Lemma I.3.16. *Let $f, g \in \Omega^{(0)}$. If $f \leq g$, then $\mathsf{sh}(f) \leq \mathsf{sh}(g)$ and $\mathsf{it}(f) \leq \mathsf{it}(g)$.*

Proof The first claim is obvious. The second is shown by induction on α. $\mathsf{it}[f, 0] = f(f(0)) \leq f(g(0)) \leq g(g(0)) = \mathsf{it}[g, 0]$. The limit case is by the continuity of $\mathsf{it}(f)$ and $\mathsf{it}(g)$. Next, we consider successors of the form $\gamma{+}1$. By I.H., $\gamma_0 := \mathsf{it}[f, \gamma] \leq \mathsf{it}[g, \gamma] =: \gamma_1$. By Lemma I.3.15 we have that $\gamma_0 \in f'$ and $\gamma_1 \in g'$. Using Lemma I.3.9, we obtain that $\mathsf{it}[f, \gamma{+}1] = f_-(\gamma_0) = f(\gamma_0{+}1) \leq g(\gamma_1{+}1) = g_-(\gamma_1) = \mathsf{it}[g, \gamma{+}1]$. Finally, we consider successors of the form $\alpha{+}2$. By I.H., $\gamma_0 := \mathsf{it}[f, \alpha{+}1] \leq \mathsf{it}[g, \alpha{+}1] =: \gamma_1$. By Lemma I.3.15 we have that $\gamma_0 \notin f'$ and $\gamma_1 \notin g'$. Hence, $\mathsf{it}[f, \alpha{+}2] = f_-(\gamma_0) = f(\gamma_0) \leq g(\gamma_1) = g_-(\gamma_1) = \mathsf{it}[g, \alpha{+}2]$. $\square$

We also have the following variant.

Lemma I.3.17. *Let $f, g \in \Omega^{(0)}$. If $f \subseteq g$, then $f' \subseteq g'$ and $\mathsf{sh}(f) \subseteq \mathsf{sh}(g)$.*

Proof Suppose that $f \subseteq g$. If $\gamma \in f'$, then $\gamma \leq g(\gamma) \leq f(\gamma) = \gamma$, thus $\gamma \in g'$. For the second claim, assume that for some γ, $f(\gamma) \notin \mathsf{sh}(g)$, that is, $f(\gamma) = g(\gamma'+n+1)$ for some n, and argue for a contradiction: since f is normal, there is some $\delta < \gamma$, so that $g(\gamma'+n) < f(\delta) < g(\gamma'+n+1)$, which contradicts $f \subseteq g$! □

Since e.g. $\mathsf{it}^{\gamma_0}(g) = \bigcap_{\xi<\gamma_0} \mathsf{it}^{\xi}(g)$, we also consider the more general situation where $f = \bigcap_{\xi<\gamma_0} f_\xi$ for a family $(f_\xi : \xi < \gamma_0)$ of normal functions, and wonder how to approximate $f(\alpha)$. As f is normal, $f_{\gamma_0}(\alpha) = \mathsf{sup}_{\beta<\alpha} f_{\gamma_0}(\beta)$ if α is a limit. Otherwise, we approximate $f_{\gamma_0}(\alpha)$ using the normal functions defined below.

Definition I.3.18. *Let $(f_\xi : \xi < \gamma_0)$ a family of normal functions. Depending on this family and a start value β, we define a continuous function $s_\beta : \gamma_0+1 \to \Omega$ as follows:*

(i) $s_\beta(0) := \beta+1$,

(ii) $s_\beta(\xi+1) := f_\xi(s_\beta(\xi))$,

(iii) $s_\beta(\gamma) := \mathsf{sup}_{\xi<\gamma} s_\beta(\xi)$.

To formulate the next lemma, we let $\mathsf{next}(g, \beta) := \mathsf{min}\{\delta \in g : \delta > \beta\}$ be the next ordinal in the range of g above β. For instance, $g(\alpha+1) = \mathsf{next}(g, \beta)$ for $\beta := g(\alpha)$.

Lemma I.3.19. *Assume that $f_{\gamma_0} := \bigcap_{\xi<\gamma_0} f_\xi$, and for each $\xi < \eta < \gamma_0$, $f_\xi \in \Omega^{(0)}$, $f_\eta \subseteq f_\xi$, and $f_{\xi+2} \subseteq f'_\xi$. Further, $s_\beta : \gamma_0+1 \to \Omega$ are the functions from Definition I.3.18. Then, we have for each β and each limit $\gamma \leq \gamma_0$,*

(i) $s_\beta(\gamma) = \mathsf{next}(f_\gamma, \beta)$,

(ii) *s_β is strictly increasing.*

In particular, $s_0(\gamma) = f_\gamma(0)$, and for $\beta := f_\gamma(\alpha)$, $s_\beta(\gamma) = f_\gamma(\alpha+1)$.

Proof (i) As $s_\beta(\xi+1) = f_\xi(s_\beta(\xi))$, we have $s_\beta(\xi+1) \in f_\xi$, and since for each $\xi' < \gamma$, $s_\beta(\gamma) = \mathsf{sup}_{\xi'<\xi<\gamma} s_\beta(\xi) \in f_{\xi'}$, we also have $s_\beta(\gamma) \in \bigcap_{\xi<\gamma} f_\xi = f_\gamma$ as f_γ is closed. As $\beta < s_\beta(\gamma)$, $s_\beta(\gamma) = \mathsf{next}(f_\gamma, \beta)$ follows, if $s_\beta(\xi) < \mathsf{next}(f_\gamma, \beta)$ for all $\xi < \gamma$, which is shown by transfinite induction on ξ. $s_\beta(0) = \beta+1 < \mathsf{next}(f_\gamma, \beta)$, and if $s_\beta(\xi) < \mathsf{next}(f_\gamma, \beta)$, then $s_\beta(\xi+1) = f_\xi(s_\beta(\xi)) < f_\xi(\mathsf{next}(f_\gamma, \beta)) = \mathsf{next}(f_\gamma, \beta)$, since $f_\gamma \subseteq f_{\xi+2} \subseteq f'_\xi$. The limit case is by the continuity of s_β.
(ii) By definition, s_β is strictly increasing, if for all $\xi \leq \gamma_0$, $s_\beta(\xi) \notin f'_\xi$. We show this by induction on ξ. Clearly, $s_\beta(0) = \beta+1 \notin f'_0$. If $s_\beta(\xi) \notin f'_\xi$, then by Lemma I.3.7, $s_\beta(\xi+1) = f_\xi(s_\beta(\xi)) \notin f'_\xi$. As $f_{\xi+1} \subseteq f_\xi$ and Lemma I.3.17 yield $f'_{\xi+1} \subseteq f'_\xi$, we also have $s_\beta(\xi+1) \notin f'_{\xi+1}$. In the limit case, we have $s_\beta(\gamma) =_{(i)} \mathsf{next}(f_\gamma, \beta)$. Therefore, $s_\beta(\gamma) = f_\gamma(\delta)$ for some $\delta \notin \mathsf{lim}(\Omega)$. So $\delta \notin f'_\gamma$, and thus $s^\beta_\gamma \notin f'_\gamma$. □

I.4 A substitute for transfinite induction

We aim to prove certain statements of the form $(\forall\alpha \in \mathsf{field}(\prec))A(\alpha)$ in T^ϵ, for instance our main result, which states that for each name x, the operation Op_x proves the functional H_x, by transfinite induction along the well-founded ordering $\prec$. We start by the following simple observation.

Lemma I.4.1. *For each* L_2*-formula* A*, if* $\mathsf{T}^\epsilon \vdash A \vee (\mathsf{ACA})$ *and* $\mathsf{ACA}_0 \vdash A$*, then* $\mathsf{T}^\epsilon \vdash A$*.*

The theorem below can be regarded as an internal variant of the following.

If for all β, $\mathsf{T}^\epsilon \vdash A(\beta)$ follows if $\mathsf{T}^\epsilon \vdash A(\alpha)$ for all $\alpha < \beta$, then for all α, $\mathsf{T}^\epsilon \vdash A(\alpha)$.

The statement $\mathsf{T}^\epsilon \vdash A(\beta)$ is approximated by the formula $\forall X(A(\beta)\restriction X)$, expressing that $A(\beta)$ holds in all models X. Therefore, the following convention is very useful: if a capital letter, say C, denote an L_2-formula, then the corresponding lower case letter c denotes the Π^1_1-formula $\forall X C\restriction X$.

Now we can state the theorem which is proved at the end of this section.

Theorem I.4.2. *Let* $A(u)$ *be an* L_2*-formulas without free set variables, and* $\prec$ *a binary relation symbol. Assume that*

(i) $\mathsf{T}^\epsilon \vdash A(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_\prec(x))$.

(ii) $\mathsf{ACA}_0 \vdash \forall x[(\forall y \prec x)a(y) \to A(x)]$.

Then, $\mathsf{T}^\epsilon \vdash (\forall x \in \mathsf{field}(\prec))A(x)$.

Usually, the theorem is applied to formulas $A(x)$ of the form $\mathsf{Op}^\varphi_x(\breve{\mathsf{T}}) \to C(x)$, for some specific $\mathsf{L}_2(\mathsf{P})$-formula $\varphi(u) := \varphi^{f,\prec,\rightsquigarrow,\vartheta}(u)$. That is, if $\neg A(x)$, then we have $\mathsf{Op}^\varphi_x(\breve{\mathsf{T}})$, which by the Representation Theorem I.2.26 entails $\mathsf{Wo}_\prec(x)$ and (ACA) (as $\breve{\mathsf{T}}$ implies (ACA) by convention, and by the Representation Theorem, $\mathsf{Op}^\varphi_x(\breve{\mathsf{T}})$ implies $\mathsf{p}_1(\breve{\mathsf{T}})$, which further implies $\mathsf{p}_1((\mathsf{ACA}))$ and (ACA)). Therefore, assumption (i) is guaranteed.

At times, we require a slight strengthening of the above theorem, namely the corollary below. There, we apply the theorem to a formula $A(u)$ of the form $B \wedge b \wedge d \to C(u)$, where $u \notin \mathsf{FV}_0(B \wedge b \wedge d)$. Due to the special form of A, we have the following.

Corollary I.4.3. *Let* $A(u) := B \wedge b \wedge d \to C(u)$ *be an* L_2*-formulas without free set variables and* $u \notin \mathsf{FV}_0(B \wedge b \wedge d)$*, and* $\prec$ *a binary relation symbol. Assume that*

(i) $\mathsf{T}^\epsilon \vdash A(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_\prec(x))$.

(ii) $\mathsf{ACA}_0 \vdash B \wedge b \wedge d \wedge (\forall y \prec x)c(y) \to C(x)$.

Then, $\mathsf{T}^\epsilon \vdash (\forall x \in \mathsf{field}(\prec))A(x)$.

Let us collect some auxiliary properties needed for the proof of the above theorem. First, note that ACA_0 proves that each set is contained in a transitive one.

Lemma I.4.4. $\mathsf{ACA}_0 \vdash \forall Z \exists X[Z \dot{\in} X \wedge \mathsf{trans}{\restriction}X]$.

Proof Having arithmetical comprehension at hand, given a set Z, we let $X := \{\langle x, s\rangle : s \in \mathsf{seq} \wedge x \in (Z)_{(s)_0,\dots,(s)_{\mathsf{lh}(s)-1}}\}$, where $\mathsf{seq} \subsetneq \mathsf{N}$ are the sequence numbers. Then $Z = (X)_{\langle\rangle}$, and if $\emptyset \neq V \dot{\in} W \dot{\in} X$ and $V = (W)_v$ and $W = (X)_s$, then $x \in V$ iff $\langle x, v\rangle \in (X)_s$ iff $\langle x, v\rangle \in (Z)_{s_0,\dots,(s)_{\mathsf{lh}(s)-1}}$ iff $x \in (Z)_{s_0,\dots,(s)_{\mathsf{lh}(s)-1},v}$ iff $x \in (X)_{s*\langle v\rangle}$. If $\emptyset \dot{\in} W \dot{\in} X$, then also $\emptyset \dot{\in} X$, as $\mathsf{seq} \subsetneq \mathsf{N}$. Technically, we define X as follows: Let h be a primitive recursive function so that $h(x, \langle\rangle) = x$ and $h(x, \langle (s)_0, \dots (s)_n, s_{n+1}\rangle) = h(\langle x, s_{n+1}\rangle, \langle (s)_0, \dots (s)_n\rangle)$. Then $X = \{\langle x, s\rangle : s \in \mathsf{seq} \wedge h(x, s) \in Z\}$. $\square$

The next lemma needs an assumption $\forall Y \exists X (Y \dot{\in} X)$ which follows readily from pair, as $Z \dot{\in} (Z + Z)$ (cf. Definition I.2.3).

Lemma I.4.5. *If A is Π^1_1, then* $\mathsf{T}^\epsilon + \mathsf{trans} \vdash A \to a$, *and* $\mathsf{T}^\epsilon + \mathsf{pair} \vdash a \to A$.

Proof Let $A = \forall Y B(Y)$ for some arithmetical $B(U)$. If A, then also $B((X)_y)$, since trans implies that $(X)_y$ is a set. Therefore a. For the second claim, fix some set Y. As there is a set X with $Y \dot{\in} X$, $B(Y)$ follows from $A{\restriction}X$. $\square$

For later reference, we also record the following.

Lemma I.4.6. *If* $\mathsf{T}^\epsilon \vdash a \wedge B \to C$, *then* $\mathsf{T}^\epsilon + \mathsf{trans} \vdash a \wedge b \to c$.

Proof If $\mathsf{T}^\epsilon \vdash a \wedge B \to C$, then also $\mathsf{T}^\epsilon \vdash \forall X(a \wedge B \to C){\restriction}X$, thus by logic, also $\mathsf{T}^\epsilon \vdash \forall X(a{\restriction}X) \wedge b \to c$. Now Lemma I.4.5 yields $\mathsf{T}^\epsilon + \mathsf{trans} \vdash a \wedge b \to c$. $\square$

Lemma I.4.7. *If A is Π^1_1, then* $\mathsf{ACA}_0 \vdash \forall X(\mathsf{trans}{\restriction}X \to A{\restriction}X) \to A$.

Proof Let $A = \forall Y D(Y)$ for some arithmetical $D(U)$. If $\forall X(\mathsf{trans}{\restriction}X \to A{\restriction}X)$, but $\neg D(Z)$ for some set Z, then by Lemma I.4.4 there were a transitive X with $Z \dot{\in} X$, thus $\neg A{\restriction}X$! $\square$

Proof [Theorem I.4.2] Assume (i) and (ii). Because of (i), it suffices to show that $\mathsf{ACA}_0 \vdash \mathsf{Wo}_\prec(x) \to A(x)$. By (ii), $\mathsf{ACA}_0 \vdash (\forall y \prec x)a(y) \to A(x)$, hence by (i), $\mathsf{T}^\epsilon \vdash (\forall y \prec x)a(y) \to A(x)$. Thus, $\mathsf{T}^\epsilon \vdash [(\forall y \prec x)a(y) \to A(x)]{\restriction}X$ by Lemma I.1.9, which further yields $\mathsf{T}^\epsilon \vdash (\forall y \prec x)a(y){\restriction}X \to A(x){\restriction}X$. Now we first quantify the X on the left side of the implication, then the X on the right, and we obtain $\mathsf{T}^\epsilon \vdash (\forall y \prec x)\forall X a(y){\restriction}X \to \forall X A(x){\restriction}X$, that is $\mathsf{T}^\epsilon \vdash (\forall y \prec x)\forall X a(y){\restriction}X \to a(x)$. Next, Lemma I.4.5 and the fact that $a(y)$ is Π^1_1 yields $\mathsf{T}^\epsilon + \mathsf{trans} \vdash (\forall y \prec x)a(y) \to a(x)$, that is, for $\mathcal{C} := \{x : a(x)\}$, $\mathsf{T}^\epsilon \vdash \mathsf{trans} \to \mathsf{Prog}_\prec(\mathcal{C})$.

Again by Lemma I.1.9, $\mathsf{T}^\epsilon \vdash \mathsf{trans}{\restriction}X \to \mathsf{Prog}_\prec(\mathcal{C}{\restriction}X)$. Since ACA_0 implies that $\mathcal{C}{\restriction}X$ is a set, $\mathsf{ACA}_0 \vdash \mathsf{Wo}_\prec(x) \to [\mathsf{trans}{\restriction}X \to (\forall y \prec x)(y \in \mathcal{C}{\restriction}X)]$. By Lemma I.4.7, $\mathsf{ACA}_0 \vdash \mathsf{Wo}_\prec(x) \to (\forall y \prec x)(y \in \mathcal{C})$, therefore $\mathsf{ACA}_0 \vdash \mathsf{Wo}_\prec(x) \to (\forall y \prec x)a(y)$. Now (ii) implies $\mathsf{ACA}_0 \vdash \mathsf{Wo}_\prec(x) \to A(x)$. $\square$

Proof [Corollary I.4.3] We show that condition (ii) of the corollary is actually equivalent to condition (ii) of the theorem, given that $A(u)$ is of the form $C_0 \to C(u)$ with $C_0 := B \wedge b \wedge d$ and $u \notin \mathsf{FV}_0(C_0)$. For each formula F, $\mathsf{ACA}_0 \vdash \forall X f{\restriction}X \leftrightarrow f$, by Lemma I.4.5. Hence, $(\forall y \prec x)a(y)$ iff $b \wedge d \to (\forall y \prec x)c(x)$, and so all the following formulas are equivalent over ACA_0

(i) $\forall x[(\forall y \prec x)a(y) \to A(x)]$,

(ii) $\forall x[(b \wedge d \to (\forall y \prec x)c(y)) \to (C_0 \to C(x))]$,

(iii) $\forall x[C_0 \to ((\forall y \prec x)c(y) \to C(x))]$,

(iv) $C_0 \to \forall x((\forall y \prec x)c(y) \to C(x))$,

(v) $C_0 \wedge (\forall y \prec x)c(y) \to C(x)$.

$\square$

Chapter II

Provable functions: the case $N_0 = 2$

In this chapter, we first introduce sets Q_2 and Q_2^H of names to address all operation $(\mathsf{Op}_x : x \in Q_2)$ and all type-2 functionals $(H_x : x \in Q_2^H)$ that play a role in the reduction process of $\mathsf{p}_3(\mathsf{ACA}_0)$. Then, we define when "$\mathsf{Op}_x$ proves H_{x^H}", and show that this holds for all $x \in Q_2$, where $\cdot^H : Q_2 \to Q_2^H$ assigns to each name x of an operation the name x^H of its corresponding functional (as a first approximation, think of x^H as the identity).

A name indicates how a functional or operation is constructed by iterated transfinite composition from the basic functionals $\mathsf{It} := \mathsf{It}_2$ and $\mathsf{it} := \mathsf{It}_1$ and the basic operations p_2 and p_1, respectively. The subscript $\cdot_2$ indicates that we restrict to names of level two; Q^H and Q will then be used to denote the respective names of all finite levels. In the sequel, we also restrict to names of arbitrary large levels N_0.

In order to give a first provisional description of what "Op_x proves H_{x^H}" is to express, we say that "T proves f", if T^ϵ proves

$$\mathsf{Prv}_0(x) := \breve{\mathsf{T}} \to \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_\mathsf{T}, \alpha) \to \mathsf{Wo}_\lhd(f(\alpha))],$$

which state that $\breve{\mathsf{T}}$ implies $\mathsf{Wo}_\lhd(f(\alpha))$ upon the assumptions $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{TI}_\lhd(\mathcal{C}_\mathsf{T}, \alpha)$, where $\mathcal{C}_\mathsf{T}$ is some class depending on x, and $\lhd$ is a primitive recursive well-ordering. "Op proves H" then states that for all T and f so that "T proves f", we also have "$\mathsf{Op}(\mathsf{T})$ proves $H(f)$".

We treat the case $N_0 = 2$ first, although this is not yet the generic case: many problems do not surface at all, or only in a trivial form, so that no extra machinery is required in order to solve them. Therefore, the goal of this chapter is not to provide the simplest possible proof, but one that neatly extends to the general case.

Names for operations and functionals are ordered sets $(Q_2, <_2)$ and $(Q_2^H, <_2^H)$. Compared to the general case, the names Q_2^H and Q_2 are quite simple and manageable.

Having names at hand will allow us to define Op_x using the Representation Theorem, and to prove "Op_x proves H_{x^H}" by transfinite induction along a suitable well-founded relation $\rightsquigarrow^*$ on Q_2.

Names are special nested sequences. Since the names Q_2^H for functionals are somewhat simpler than the names Q_2 for operations, we first have a brief look at Q_2^H. $q_0 := \langle\rangle$ is the only name of level 0, and $\langle(\alpha, q_0)\rangle$ with $0 < \alpha$ are the names of level one, and $(\alpha, q_0) <_2^H (\beta, q_0)$ iff $\alpha < \beta$. Names of level two may be sequences of length bigger than one: if $x_1 < \ldots < x_k$ are names of level one and $0 < \alpha_i$ $(1 \leq i \leq k)$, then $\langle(\alpha_1, x_1), \ldots (\alpha_k, x_k)\rangle$ is a name of level two. Functionals of type-2 are then named in the following manner: $H_{(\alpha,q_0)} := \mathsf{it}^\alpha$, $H_{(\beta,(\alpha,q_0))} := (\mathsf{It}^\alpha(\mathsf{it}))^\beta$, and if $\langle x_1, \ldots, x_k\rangle$ is a name of level two with $k > 1$, then $H_{\langle x_1,\ldots,x_k\rangle} := H_{x_1} \circ \ldots \circ H_{x_k}$.

Each $x \in Q_2^H$ also names an operation Op_x. The naming schema for operations is similar, however, there are some differences that are discussed in the next section. In particular, $Q_2^H \subsetneq Q_2$, since we need more names for operations than for functionals. We still have that $\mathsf{Op}_{(\alpha,q_0)}$ iff p_1^α, and $\mathsf{Op}_{(n,(\alpha,q_0))}$ iff $(\mathsf{p}_2^\alpha\mathsf{p}_1)^n$. But e.g. $\mathsf{Op}_{(\gamma,(\alpha,q_0))}$ iff $(\mathsf{p}_1\mathsf{p}_2^\alpha\mathsf{p}_1)^\gamma$ and $\mathsf{Op}_{(\gamma+n+1,(\alpha,q_0))}$ iff $(\mathsf{p}_2^\alpha\mathsf{p}_1)^{\gamma+n}$.

The organization of this chapter is as follows. First, we introduce names for operations and functionals. Then, we have a closer look at names and define approximations and normal forms, which leads to the definition of the relations $\rightsquigarrow$ and $\rightsquigarrow^*$, and a formal definitions of Op_x by means of the Representation Theorem. After collecting relevant properties of the functionals H_x and the operations Op_x, we are then in position to show that for each $x \in Q_2^* := Q_2 \setminus \{q_0\}$, "$\mathsf{Op}_x$ proves H_{x^H}".

Differences between operations and functionals

In this section, we (still somewhat informal) explain in what sense operations and functionals behave differently. We start by reviewing some basic properties of operations that are constantly used in the sequel. Recall that $\mathsf{Op} \Rightarrow \mathsf{Op}'$ states that for all $\breve{\mathsf{T}}$, $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}'(\breve{\mathsf{T}})$.

Firstly, recall (cf. Definition I.2.1) that if $\breve{\mathsf{T}}$ and $\breve{\mathsf{T}}'$ are open L_2-sentences so that $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$, then also $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}')$ for each operation Op. In particular, since trivially $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \wedge \breve{\mathsf{T}}' \to \breve{\mathsf{T}}$ and $\mathsf{T}^\epsilon \vdash \forall x\breve{\mathsf{T}}(x) \to \breve{\mathsf{T}}(u)$ $(u \notin \mathsf{FV}(\forall x\breve{\mathsf{T}}))$, we have that $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}} \wedge \breve{\mathsf{T}}') \to \mathsf{Op}(\breve{\mathsf{T}})$ and $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\forall x\breve{\mathsf{T}}(x)) \to \forall x\mathsf{Op}(\breve{\mathsf{T}}(x))$.

Secondly, as (ACA) is Π_2^1, Lemma I.2.12 yields that $\mathsf{T}^\epsilon \vdash \mathsf{p}_1((\mathsf{ACA})) \to (\mathsf{ACA})$. Thus, as $\mathsf{p}_2 \Rightarrow \mathsf{p}_1$, also $\mathsf{T}^\epsilon \vdash \mathsf{p}_2((\mathsf{ACA})) \to (\mathsf{ACA})$. Since p_1 is an operation, we obtain $\mathsf{T}^\epsilon \vdash \mathsf{p}_1\mathsf{p}_2((\mathsf{ACA})) \to \mathsf{p}_1(\mathsf{ACA})$, and further, as $\mathsf{T}^\epsilon \vdash \mathsf{p}_1((\mathsf{ACA})) \to (\mathsf{ACA})$, we conclude that also $\mathsf{T}^\epsilon \vdash \mathsf{p}_1\mathsf{p}_2((\mathsf{ACA})) \to (\mathsf{ACA})$.

Thirdly, if $\breve{\mathsf{T}}$ is Π_2^1, then $\mathsf{T}^\epsilon \vdash \mathsf{p}_2(\breve{\mathsf{T}}) \leftrightarrow \mathsf{p}_2\mathsf{p}_1(\breve{\mathsf{T}})$. To see this, we work informally in

T^ϵ and assume $\mathsf{p}_2(\breve{\mathsf{T}})$. Then we have also $\mathsf{p}_1(\breve{\mathsf{T}})$, which is Π^1_2. Using Lemma I.2.14 yields $\mathsf{p}_2\mathsf{p}_1(\breve{\mathsf{T}})$. Conversely, as $\breve{\mathsf{T}}$ is Π^1_2, $\mathsf{T}^\epsilon \vdash \mathsf{p}_1(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}$, thus, as p_2 is an operation, also $\mathsf{T}^\epsilon \vdash \mathsf{p}_2\mathsf{p}_1(\breve{\mathsf{T}}) \to \mathsf{p}_2(\breve{\mathsf{T}})$.

Next, we elaborate on some differences of the operation p_2 and the functional It. A first thing to observe is that we can *compose* the functional It with any other type-3 functional F to obtain the type-3 functional $(\mathsf{It} \circ F)$. Alternatively, we can *apply* the type-3 functional It to a type-2 functional G to obtain a type-2 functional $\mathsf{It}(G)$. However, we cannot apply It to F, or compose It with G. For the operation p_2 there is no such distinction: $\mathsf{p}_2 \circ \mathsf{p}_2$ and $\mathsf{p}_2 \circ \mathsf{p}_1$ are both well-defined compositions of operations.

Example II.0.8. *Recall that* $\mathsf{p}_2(\mathsf{ACA}_0)$ *is* Σ^1_1-DC_0 *and that* $\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0)$ *is* ATR_0 *(cf. Lemma I.2.10). Hence,* $\mathsf{p}_2(\mathsf{p}_2(\mathsf{ACA}_0))$ *is a theory that claims* Π^1_2*-reflection on* ω*-models of* Σ^1_1-DC_0*.* $\mathsf{p}_2(\mathsf{p}_2(\mathsf{ACA}_0))$ *is equivalent to a theory with* Σ^1_1*-transfinite dependent choice (cf. Rüede [19]) of ordinal strength* $\varphi\omega 00$ *(cf. Rüede [20]).*

Next, we look at the theory $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$*, analyzed in Jäger and Strahm [8] and shown to have ordinal strength* $\varphi 1\omega 0$*. We see that* $\mathsf{p}_2(\mathsf{ATR}_0)$ *is* $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$*. Since* $\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ *implies* (ACA)*,* $\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ *implies* $\mathsf{p}_2((\mathsf{ACA}))$*, i.e.* $(\Sigma^1_1\text{-}\mathsf{DC})$*. And* $\mathsf{p}_2(\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0))$ *proves* $\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ *by Lemma I.2.12. Conversely, since* $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$ *proves* $\mathsf{p}_2((\mathsf{ACA})) \wedge \mathsf{p}_1(\mathsf{p}_2((\mathsf{ACA}))$*, and* $\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ *is* Π^1_2*, we obtain, using Lemma I.2.14,* $\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$*.*

Summing up, we have the following.

(i) $\mathsf{p}_2(\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0))$ *is a theory of strength* $\varphi 1\omega 0$*,*

(ii) $\mathsf{p}_2^2(\mathsf{ACA}_0)$ *is a theory of strength* $\varphi\omega 00$*.*

Since $\mathsf{p}_2((\mathsf{ACA}))$ *iff* $\mathsf{p}_2\mathsf{p}_1((\mathsf{ACA}))$*, we can present these theories also in the following slightly different way, which immediately reveals its connection the the corresponding functionals. Namely,* $\mathsf{p}_2^2(\mathsf{ACA}_0)$ *is* $\mathsf{p}_2^2\mathsf{p}_1(\mathsf{ACA}_0)$*, and according to the above discussion,* $\mathsf{ATR}_0 + (\Sigma^1_1\text{-}\mathsf{DC})$ *is* $(\mathsf{p}_2\mathsf{p}_1)(\mathsf{ATR}_0)$ *is* $(\mathsf{p}_2\mathsf{p}_1)(\mathsf{p}_2\mathsf{p}_1)(\mathsf{ACA}_0)$*. Indeed, it turns out that* $\mathsf{p}_2^2\mathsf{p}_1$ *corresponds to* $\mathsf{It}^2(\mathsf{it})$*, and that* $(\mathsf{p}_2\mathsf{p}_1)^2$ *corresponds to* $(\mathsf{It}(\mathsf{it}))^2$*.*

This suggests that "p_2 *applies to* p_1*" and that "*p_2 *composes with* p_2*". To keep this distinction visible, we build our operations from the components* $\mathsf{p}_2^{1+\alpha}\mathsf{p}_1$ *and* p_1*. This corresponds to the situation with functionals: the functionals with names in* $Q_2^H \setminus \{q_0\}$ *are build from components* $\mathsf{It}^{1+\alpha}(\mathsf{it})$ *and* it*.*

Next, we explain why we need more names for operations than for functionals. For this discussion, we work informally in T^ϵ. Observe that $\mathsf{p}_2^\omega\mathsf{p}_1(\breve{\mathsf{T}})$ implies $\mathsf{Op}(\breve{\mathsf{T}})$ for $\mathsf{Op} := \forall n(\mathsf{p}_1\mathsf{p}_2^n\mathsf{p}_1)$ (the operation which maps $\breve{\mathsf{T}}$ to $\forall n(\mathsf{p}_1\mathsf{p}_2^n\mathsf{p}_1(\breve{\mathsf{T}}))$). As $\mathsf{Op}(\breve{\mathsf{T}})$ is Π^1_2, m-fold use of Lemma I.2.14 yields $(\forall n(\mathsf{p}_1\mathsf{p}_2^n\mathsf{p}_1))^m(\breve{\mathsf{T}})$ for each $m \in \mathbb{N}$. Obviously, the operation Op is different from $\mathsf{p}_2^\omega\mathsf{p}_1$. On the other hand, one readily obtains that $\mathsf{It}^{n+1}(\mathsf{it}) \subseteq \mathsf{it} \circ \mathsf{It}^n(\mathsf{it}) \subseteq \mathsf{It}^n(\mathsf{it})$ (cf. Lemma II.1.6 and Convention I.3.5), and hence $H := \bigcap_n(\mathsf{it} \circ \mathsf{It}^n(\mathsf{it})) = \mathsf{It}^\omega(\mathsf{it})$. While we therefore need no extra name for the functional H, an extra name is needed for the "corresponding" operation Op. We pick $(1,(\omega,q_0)^-)$ as a name for Op, i.e. $\forall n(\mathsf{p}_1\mathsf{p}_2^n\mathsf{p}_1)$ (recall that $\mathsf{Op}_{(1,(\omega,q_0))}$ is $\mathsf{p}_2^\omega\mathsf{p}_1$ and $H_{(1,(\omega,q_0))}$ is $\mathsf{It}^\omega(\mathsf{it})$).

As expected, it turns out that "$\forall n(\mathsf{p}_1\mathsf{p}_2^n\mathsf{p}_1)$ proves $\mathsf{It}^\omega(\mathsf{it})$", in other words, we have that "$\mathsf{Op}_{(1,(\omega,q_0)^-)}$ proves $H_{(1,(\omega,q_0))}$". Further, we will see that "$\mathsf{p}_2^\omega\mathsf{p}_1$ proves $\mathsf{It}^{\omega+1}(\mathsf{it})$", that is, "$\mathsf{Op}_{(1,(\omega,q_0))}$ proves $H_{(1,(\omega+1,q_0))}$". So the correspondence is slightly skewed. We do not have that Op_x is the counterpart of H_x; only "Op_x proves H_{x^H}", where map $\cdot^H : Q_2 \to Q_2^H$ (cf. Definition II.2.17) restores the correspondence. For instance, $(1,(\omega,q_0)^-)^H = (1,(\omega,q_0))$, and $(1,(\omega,q_0))^H = (1,(\omega{+}1,q_0))$.

Next, we explain why $(\omega,(1,q_0))$ is a name for the functional $(\mathsf{It}(\mathsf{it}))^\omega$, but only $(\omega{+}1,(1,q_0))$ is a name for the operation $(\mathsf{p}_2\mathsf{p}_1)^\omega$. To begin with, we anticipate that for each n, "$(\mathsf{p}_2\mathsf{p}_1)^n$ proves $(\mathsf{It}(\mathsf{it}))^n$". Hence, it is plausible that "$\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$ prove $\bigcap(\mathsf{it} \circ \mathsf{It}(\mathsf{it}))^n$". However, $\bigcap(\mathsf{it} \circ \mathsf{It}(\mathsf{it}))^n$ is $(\mathsf{It}(\mathsf{it}))^\omega$. So "$\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$ prove $(\mathsf{It}(\mathsf{it}))^\omega$", and we thus use $(\omega,(1,q_0))$ as a name for the operation $\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$. As shown in the next Lemma, the operation $(\mathsf{p}_2\mathsf{p}_1)^\omega$ is stronger.

Lemma II.0.9. $(\mathsf{p}_2\mathsf{p}_1)^\omega \Leftrightarrow (\mathsf{p}_2\mathsf{p}_1) \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$.

Proof We work informally in T^ϵ. For each n, $(\mathsf{p}_2\mathsf{p}_1)^\omega(\breve{\mathsf{T}})$ implies $(\mathsf{p}_2\mathsf{p}_1)(\mathsf{p}_2\mathsf{p}_1)^n(\breve{\mathsf{T}})$ which implies $\mathsf{p}_1(\mathsf{p}_2\mathsf{p}_1)^n(\breve{\mathsf{T}})$ by Lemma I.2.12. Thus, $(\mathsf{p}_2\mathsf{p}_1)^\omega \Rightarrow \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$. So $(\mathsf{p}_2\mathsf{p}_1)^\omega(\breve{\mathsf{T}})$ implies $\mathsf{p}_2(\mathsf{p}_1(\breve{\mathsf{T}})) \wedge \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n(\breve{\mathsf{T}}))$. Now $(\mathsf{p}_2 \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n)(\breve{\mathsf{T}})$ follows by Lemma I.2.14. Conversely, as $\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$ is Π^1_2, $(\mathsf{p}_2 \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n)(\breve{\mathsf{T}})$ implies $(\mathsf{p}_2\mathsf{p}_1 \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n)(\breve{\mathsf{T}})$, which in turn implies for each n, $((\mathsf{p}_2\mathsf{p}_1) \circ (\mathsf{p}_2\mathsf{p}_1)^n)(\breve{\mathsf{T}})$. Hence, $(\mathsf{p}_2\mathsf{p}_1)^\omega \Leftrightarrow \mathsf{p}_2 \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n \Leftrightarrow (\mathsf{p}_2\mathsf{p}_1) \circ \forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$. □

Having fixed $(\omega,(1,q_0))$ as a name for the operation $\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$, the above lemma states that $(\mathsf{p}_2\mathsf{p}_1)^\omega$ iff $\mathsf{Op}_{(1,(1,q_0))} \circ \mathsf{Op}_{(\omega,(1,q_0))}$. Further, it turns out that "$(\mathsf{p}_2\mathsf{p}_1)^\omega$ proves $(\mathsf{It}(\mathsf{it}))^{\omega+1}$". Since, $(\mathsf{It}(\mathsf{it}))^{\omega+1}$ is $H_{((1,(1,q_0))} \circ H_{((\omega,(1,q_0))}$, which goes by the name $H_{((\omega+1,(1,q_0))}$, it makes sense to assign $(\mathsf{p}_2\mathsf{p}_1)^\omega$ the name $(\omega{+}1,(1,q_0))$.

More generally, we have the following Lemma, which can be proved analogously to the previous one. We will give a proof later, using the proper definition of Op_x (cf. Theorem II.4.20).

Lemma II.0.10.

(i) $(\mathsf{p}_2\mathsf{p}_1)^\gamma \Leftrightarrow (\mathsf{p}_2\mathsf{p}_1)(\forall\xi < \gamma)(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^\xi$, *and*

(ii) $(\mathsf{p}_2\mathsf{p}_1)^{\gamma+n+1} \Leftrightarrow (\mathsf{p}_2\mathsf{p}_1)^n(\forall\xi < \gamma)(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^\xi$.

A further thing to keep in mind is that the Representation Theorem should yield a formula $\varphi(u)$ that represents Op_x for each name x, (that is, $\varphi(u)\{\breve{\mathsf{T}}\restriction U\}$ iff $\mathsf{Op}_x(\breve{\mathsf{T}})$), where we regard just p_1 and p_2 as basic operations. Using Lemma II.0.9, we can regard $(\mathsf{p}_2\mathsf{p}_1)^\omega$ as $\mathsf{p}_2(\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n)$, and $\forall n(\mathsf{p}_1\mathsf{p}_2\mathsf{p}_1)^n$ is $\forall n\mathsf{p}_1(\mathsf{p}_2\mathsf{p}_1)^n$, which allows us to define $(\mathsf{p}_2\mathsf{p}_1)^\omega$ using the Representation Theorem form the basic operations p_1 and p_2 (cf. Definition II.4.1).

II.1 Names

In this section, we introduce names. Q_2 and Q_2^H are then instances of this definition, as are Q and Q^H. In order to keep subsequent definitions as simple as possible, we make use of the following notions.

For a set X, we denote by $X^{<\omega}$ the set of finite sequences with elements from X. Such a sequence of length n is usually depicted by $\langle x_1, \ldots, x_n\rangle$, and if $\sigma := \langle x_1, \ldots, x_n\rangle$, then $(\sigma)_i := x_i$ $(1 \leq i \leq n)$. The function $\mathsf{lh} : X^{<\omega} \to \mathbb{N}$ returns the length of such a sequence, and $\langle\rangle$ stands for the empty sequence with length 0. Further, $x * y$ denotes the concatenation of the finite sequences x and y.

An ordered set is a pair $(X, <)$, consisting of a set and a strict and total ordering $<$ on X. Occasionally, we consider orderings $<$ with $X \subsetneq \mathsf{field}(<)$, in which case $(X, <)$ is short for $(X, <\restriction X)$, where $<\restriction X := \{(x, y) \in X \times X : x < y\}$.

Definition II.1.1. *Let $(X, <)$ be an ordered set. Then $<_{lex}$ is the least ordering on $X^{<\omega}$ with the following properties: for all $\sigma, \sigma', \tau \in X^{<\omega}$, and $x, y \in X$,*

(i) if $\langle\rangle \neq \sigma$, then $\tau <_{lex} \sigma * \tau$,

(ii) if $x < y$, then $\sigma * \langle x\rangle * \tau <_{lex} \sigma' * \langle y\rangle * \tau$.

Hence, if there is a first position form the right where the two sequences differ, then the sequence with the $<$-bigger element at this position is the $<_{\mathrm{lex}}$-larger one. And if there is no such position, then the longer sequence is the $<_{\mathrm{lex}}$-larger one.

II.1.1 Names for functionals

In this subsection, we reiterate, now in a more formal manner, what we have outlined at the beginning of this chapter about names for functionals.

Definition II.1.2 (Names for functionals)**.** *Let $q_0 := \langle\rangle$ and $Q_0^H := \{q_0\}$.*

*(i) $(Q_1^H, <_1^H)$ is the ordered set with $Q_1^H := Q_0^H \cup \{\langle(\alpha, q_0)\rangle : 0 < \alpha\}$.
For $x, y \in Q_1^H$, we have*

$$x <_1^H y :\Leftrightarrow (x = q_0 \wedge x \neq y) \vee \exists\alpha, \beta[\alpha < \beta \wedge x = \langle(\alpha, q_0)\rangle \wedge y = \langle(\beta, q_0)\rangle].$$

(ii) $(Q_2^H, <_2^H)$ *is the ordered set with*

$$Q_2^H := \{\langle(\alpha_1, x_1), \ldots, (\alpha_k, x_k)\rangle : \vec{x} \in Q_1^H, x_1 <_1^H \ldots <_1^H x_k\},$$

and $<_2^H := \lessdot_{lex}$*, where* $\lessdot$ *orders* $(\Omega \setminus \{0\}) \times Q_1^H$*: for* $x, y \in Q_1^H$ *and ordinals* $0 < \alpha, \beta < \Omega$*,* $q_0 \lessdot (\alpha, x)$*, and* $(\alpha, x) \lessdot (\beta, y)$ *iff* $x <_1^H y \vee (x = y \wedge \alpha < \beta)$*.*

We point out that $q_0 \in Q_2^H$ *(i.e. in the definition of the set* Q_2^H*,* $\vec{x}$ *may be empty).*

As names are used quite frequently in the sequel, we stick to the following abbreviations and the next convention in order to increase readability.

Convention II.1.3. *If the context indicates that we work with names, then we write* (α, x) *for* $\langle(\alpha, x)\rangle$*. So we write e.g.* $(\alpha, (\beta, q_0))$ *for* $\langle(\alpha, \langle(\beta, q_0)\rangle)\rangle$*. Further, if* f *is a function defined on names, then we write* $f(\alpha, x)$ *for* $f(\langle(\alpha, x)\rangle)$*. Moreover, we read* $(0, q_0)$ *as the name* q_0*.*

Definition II.1.4. *We utilize* q_1*,* q_1^α *and* q_2*, to denote the following names.* $q_1 := (1, q_0)$*,* $q_2 := (1, q_1)$ *and* $q_1^\alpha := (\alpha, q_0)$*.*

Next, we assign to each $x \in Q_1^H$ a type-3 functional H_x^+, and to each name $x \in Q_2^H$ a type-2 H_x functional as follows.

Definition II.1.5. H_{q_0} *is the identity on* $\Omega^{(0)}$*,* $H_{q_0}^+$ *is the identity on* $\Omega^{(1)}$*, and for* $0 < \alpha, \beta < \Omega$ *and* $k > 1$*,*

(i) $H_{(\alpha, q_0)} := \mathsf{it}^\alpha$*,* $H_{(\alpha, q_0)}^+ := \mathsf{It}^\alpha$*,*

(ii) $H_{(\beta, x)} := (H_x^+(\mathsf{it}))^\beta$ *(*$x \neq q_0$*),*

(iii) $H_{\langle x_1, \ldots, x_k\rangle} := H_{x_1} \circ \ldots \circ H_{x_k}$*.*

Further, if $f \in \Omega^{(0)}$*, then* $f_x := H_x(f)$*.*

The general form of a name $x \in Q_2^H$ for a functional is hence $(\alpha_0, q_0) * y$ for $y := \langle(1{+}\alpha_1, (1{+}\beta_1, q_0)), \ldots, (1{+}\alpha_k, (1{+}\beta_k, q_0))\rangle$ with $\beta_1 < \ldots < \beta_k$, and the corresponding functional is $H_x = \mathsf{it}^{\alpha_0} \circ (\mathsf{It}^{1+\beta_1}(\mathsf{it}))^{1+\alpha_1} \circ \ldots \circ (\mathsf{It}^{1+\beta_k}(\mathsf{it}))^{1+\alpha_k}$.

Lemma II.1.6. *For each* $x \in Q_2^H \setminus \{q_0\}$ *and each* $y \in Q_1^H \setminus \{q_0\}$*,*

(i) $H_x \in \Omega^{(1)}$ *and* $H_y^+ \in \Omega^{(2)}$*,*

(ii) $H_x \subseteq \mathsf{it}$ *and* $H_y^+ \subseteq \mathsf{It}$*.*

Proof Recall that if $F \in \Omega^{(n+1)}$, then F is strictly inclusive, that is, $F(G) \subseteq G$ for each $G \in \Omega^{(n)}$, and further, $F^{1+\alpha} \in \Omega^{(n+1)}$ for each α (cf. Lemma I.3.12). Moreover, $F^{1+\alpha} \subseteq F$, as is readily seen by induction on α, and if $\beta \leq \alpha$, then $F^\alpha \subseteq F^\beta$. Hence, both claims are obvious if $y \in Q_1^H \setminus \{q_0\}$, as then, $y = (1+\beta, q_0)$ and $H_x^+ = \mathsf{It}^{1+\beta}$, and $\mathsf{It} \in \Omega^{(2)}$ by Corollary I.3.13.

For $x \in Q_2^H \setminus \{q_0\}$, the two claims are shown simultaneously by induction on the build up of Q_2^H. $H_{(1+\alpha,q_0)} = \mathsf{it}^{1+\alpha} \in \Omega^{(1)}$ and $\mathsf{it}^{1+\alpha} \subseteq \mathsf{it}$. If $y \neq q_0$ and $(1+\alpha, y) \in Q_2^H \setminus \{q_0\}$, then $y \in Q_1^H \setminus \{q_0\}$ and $H_y^+ \in \Omega^{(2)}$ and $H_y^+ \subseteq \mathsf{It}$ by (i), hence $H_y^+(\mathsf{it}) \in \Omega^{(1)}$ and

$$H_{(1+\alpha,y)} = (H_y^+(\mathsf{it}))^{1+\alpha} \subseteq H_y^+(\mathsf{it}) \subseteq \mathsf{It}(\mathsf{it}) \subseteq \mathsf{it}.$$

And if both claims hold for x_k, and $x = \langle x_1, \ldots, x_k \rangle$ and $k > 1$, then $H_{\langle x_1,\ldots,x_k\rangle} = H_{x_1} \circ \ldots \circ H_{x_k} \subseteq H_{x_k} \subseteq \mathsf{it}$, and $H^+_{\langle x_1,\ldots,x_k\rangle} \in \Omega^{(1)}$ as $\Omega^{(1)}$ is closed under composition. $\square$

Lemma II.1.7. *For each $x \in Q_2^H \setminus \{q_0\}$, $\mathsf{It}(H_x) \subseteq \mathsf{it} \circ H_x$.*

Proof $\mathsf{It}[H_x, f, \alpha] \in (H_x)^{2+\alpha}(f) \subseteq (H_x)^2(f) \subseteq_{L.II.1.6} (\mathsf{it} \circ H_x)(f)$. $\square$

Finally, the following is readily observed.

Lemma II.1.8. *For each $x \in Q_1^H$, $H_{(\alpha,x)} \circ H_{(\beta,x)} = H_{(\beta+\alpha,x)}$.*

II.1.2 Names for operations

We have already discussed that we need different names for the operations $\mathsf{p}_2^\gamma \mathsf{p}_1$ and $(\forall \xi < \gamma)(\mathsf{p}_1 \mathsf{p}_2^{1+\xi} \mathsf{p}_1)$, and that we plan to use $(1, (\gamma, q_0))$ as a name for the former, and $(1, (\gamma, q_0)^-)$ as a name for the latter operation. The following auxiliary definition helps us to implement such a naming schema.

Definition II.1.9. *Given a set X, then we denote by X^- a disjoint copy of X, and by $\iota : X^- \to X$ a corresponding bijection. If $(X, <)$ is an ordered set and $Y \subseteq X^-$, then $j : X \cup Y \to X$, $j(x) := x$ if $x \in X$, and $j(y) := \iota(y)$ if $y \in Y$. $(X \cup Y, <)$ is the ordered set with $\mathsf{field}(<) = X \cup Y$, and*

$$z < z' :\Leftrightarrow j(z) < j(z') \vee (j(z) = j(z') \wedge z \in Y \wedge z' \in X).$$

Further, we write x^- for $\iota^{-1}(x)$.

The set of names Q_2 for operations is a superset of the names Q_2^H for functionals. Names of level two are now not only formed using names $\vec{x} \in Q_1$ of level one, but also so-called prenames, elements of Q_1^- of the form $(\gamma, q_0)^-$.

Definition II.1.10 (Names for operations)**.** *$q_0 := \langle\rangle$ and $Q_0 := \{q_0\}$, and* *(i)* $(Q_1, <_1) := Q_0 \cup \{\langle(\alpha, q_0)\rangle : 0 < \alpha\}$ *and* $P_1 := \{(\gamma, q_0)^- : \gamma \in \mathsf{Lim}(\Omega)\} \subseteq Q_1^-$*. Now* $(Q_1 \cup P_1, <_1)$ *is the ordered set explained by Definition II.1.9.*

(ii) $(Q_2, <_2)$ *is the ordered set with*

$$Q_2 := \{\langle(\alpha_1, v_1), \ldots (\alpha_k, v_k)\rangle : \vec{v} \in P_1 \cup Q_1, v_1 <_1 \ldots <_1 v_k\},$$

and $<_2 := \lessdot_{lex}$*, where* $\lessdot$ *orders* $(\Omega\setminus\{0\}) \times Q_1 \cup P_1$ *as follows: for* $v, w \in Q_1 \cup P_1$ *and ordinals* $0 < \alpha, \beta$*, we have* $q_0 \lessdot (\alpha, v)$*, and further,* $(\alpha, v) \lessdot (\beta, w)$ *iff* $v <_1 w \vee (v = w \wedge \alpha < \beta)$*.*

Further, $Q_1^* := Q_1 \setminus \{q_0\}$ *and* $Q_2^* := Q_2 \setminus \{q_0\}$*. Again,* $Q_1 \subseteq Q_2$ *(i.e. in the definition of the set* Q_2*,* $\vec{v}$ *may be empty).*

Note that if $z^- \in P_1$, then z is the $<_2$-least element above z^-. Further, it is readily seen that $(Q_2, <_2)$ and $(Q_2 \cup P_1, <_2)$ are well-orderings.

Convention II.1.11. *We let* x, y, z *range over* Q_2*, and* v, w *over* $Q_2 \cup P_1$*. If* f *is a function defined on names, then we write* $f(\alpha, x)$ *for* $f(\langle(\alpha, x)\rangle)$*. For instance, we write* $(\gamma, q_0)^-$ *for* $\langle(\gamma, q_0)\rangle^-$*.*

Provisional definitions of the operations $(\mathsf{Op}_x : x \in Q_2^*)$

Below, we assign to each $x \in Q_2^*$ an operation Op_x, and to each $x \in Q_1^*$ an operation Op_x^+. The definition is provisional, as this assignment is semantical: given $x \in Q_2^*$, Op_x is an operation, and it is assumed that we can represent this operation by an $\mathsf{L}_2(\mathsf{P})$-sentence, by using some way to code x as a natural number. Later, this assignment is superseded by Definition II.4.1, the proper definition of the operations $(\mathsf{Op}_x : x \in Q_2^*)$ and $(\mathsf{Op}_x^+ : x \in Q_1^*)$, which provides $\mathsf{L}_2(\mathsf{P})$-formulas $\varphi(u)$ and $\varphi^+(u)$ so that Op_x^φ represents Op_x, and $\mathsf{Op}_x^{\varphi^+}$ represents Op_x^+. There, it is assumed that we have a primitive recursive relation which codes Q_2^*, which is also denoted by Q_2^*.

Definition II.1.12. *For all* $0 < \alpha, \beta < \Omega$*, and* $k > 1$*,*

(i) $\mathsf{Op}_{(\alpha, q_0)} := \mathsf{p}_1^\alpha$ *and* $\mathsf{Op}^+_{(\alpha, q_0)} := \mathsf{p}_2^\alpha$*,*

(ii) $\mathsf{Op}_{(n,(\alpha,q_0))} := (\mathsf{p}_2^\alpha \mathsf{p}_1)^n$ *and* $\mathsf{Op}_{(\gamma+n+1,(\alpha,q_0))} := (\mathsf{p}_2^\alpha \mathsf{p}_1)^{\gamma+n}$*,*

(iii) $\mathsf{Op}_{(\gamma,(\alpha,q_0))} := (\mathsf{p}_1 \mathsf{p}_2^\alpha \mathsf{p}_1)^\gamma$ *and* $\mathsf{Op}_{(\alpha,(\gamma,q_0)^-)} := ((\forall \xi < \gamma)(\mathsf{p}_1 \mathsf{p}_2^{1+\xi} \mathsf{p}_1))^\alpha$*,*

(iv) $\mathsf{Op}_{\langle x_1,\ldots,x_k\rangle} := \mathsf{Op}_{x_1} \circ \ldots \circ \mathsf{Op}_{x_k}$*.*

We see that for all $x \in Q_1^*$,

$$\mathsf{Op}_{(n,x)} \Leftrightarrow (\mathsf{Op}_x^+ \mathsf{p}_1)^n \text{ and } \mathsf{Op}_{(\gamma+n+1,x)} \Leftrightarrow (\mathsf{Op}_x^+ \mathsf{p}_1)^{\gamma+n} \text{ and } \mathsf{Op}_{(\gamma,x)} \Leftrightarrow (\mathsf{p}_1 \mathsf{Op}_x^+ \mathsf{p}_1)^{\gamma},$$

which matches almost (cf. discussion prior to Lemma II.0.10) the corresponding Definition II.1.5 for functionals. Further, we also have the following lemma which is the counterpart of Lemma II.1.8. A proof (cf. Lemma II.4.4 (iv)) is only given after the proper definition of Op_x is presented.

Lemma II.1.13. *For each $v \in Q_1 \cup P_1$ and all $0 \lhd \alpha, \beta$, $\mathsf{Op}_{(\alpha,v)} \circ \mathsf{Op}_{(\beta,v)} \Leftrightarrow \mathsf{Op}_{(\beta+\alpha,v)}$.*

II.2 Approximations and normal forms

In this section, we have a closer look at the names in Q_2, which we use to name operations. In particular, we define normal forms and two kinds of approximations, $x[\alpha]$ and $x(\alpha)$. Finally, we define $\cdot^H : Q_2 \to Q_2^H$ so that Op_x corresponds to H_{x^H}.

Before we hint at a relevant property of the approximation $x[\alpha]$ in the next paragraph, we give the definition of degree $\deg(x)$ and ordinal $o(x)$ of a name right away. A look at Definition II.1.12 then readily confirms that $\mathsf{Op}_x(\breve{\mathsf{T}})$ is Π^1_{m+2} if $\deg(x) = m{+}1$ (where $m \in \{0, 1\}$). The role of $o(x)$ will become clearer later. For now, just note that for names x with $\deg(x) = 1$, we have $o(x) = 1$ if x is a successor, and $o(x) \in \mathsf{Lim}(\Omega)$ if x is a limit w.r.t. $(Q_2, <_2)$.

Definition II.2.1. *For $x \in Q_2$ and $f \in \{\deg, o\}$, we let $f(x) := f((x)_0)$ and $f(\alpha{+}1, v) := f(1, v)$. Further,*

(i) $\deg(q_0) := 0$, $\deg(1, x^-) := 1$, $\deg(1, x) := \deg(x){+}1$ *and* $\deg(\gamma, v) := 1$.

(ii) $o(q_0) := 1$, $o(1, x^-) := o(x)$, $o(1, x) := o(x)$ *and* $o(\gamma, v) := \gamma$.

We extend $\deg$ *and o to $Q_2 \cup P_1$ by setting,* $\deg(x^-) := 0$ *and* $o(x^-) := o(x)$.

A key property of the approximation $x[\alpha]$ is the following (cf. Lemma II.4.2), which yields to a definition of Op_x by means of the Representation Theorem.

(i) If $o(x) = 1$ and $\deg(x) = m{+}1$, then $\mathsf{Op}_x \Leftrightarrow \mathsf{p}_{m+1} \circ \mathsf{Op}_{x[0]}$,

(ii) if $o(x) = \gamma$ and $\deg(x) = m{+}1$, then $\mathsf{Op}_x \Leftrightarrow (\forall \alpha < \gamma)(\mathsf{p}_{m+1} \circ \mathsf{Op}_{x[\alpha]})$.

Let us illustrate this with an example. If $x := (1, (2, q_0))$ and $y := (1, (\omega, q_0))$, then $x[0] = (1, (1, q_0))$ and $y[n] = (1, (n, q_0))$, and by Definition II.1.12, $\mathsf{Op}_{x[0]}$ iff $\mathsf{p}_2 \mathsf{p}_1$, and $\mathsf{Op}_{y[n]}$ iff $\mathsf{p}_2^{n+1} \mathsf{p}_1$. Now a) and b) (see next page) are instances of (i) and (ii), respectively.

a) Op_x iff $\mathsf{p}_2^2\mathsf{p}_1$ iff $\mathsf{p}_2 \circ \mathsf{p}_2\mathsf{p}_1$,

b) Op_y iff $\mathsf{p}_2^\omega\mathsf{p}_1$ iff $(\forall n < \omega)(\mathsf{p}_2 \circ \mathsf{p}_2^{n+1}\mathsf{p}_1)$.

Next, we define a partial function $\circ : Q_2 \times Q_2 \to Q_2$, so that if $x \circ y$ is defined, then $\mathsf{Op}_{x \circ y} \Leftrightarrow \mathsf{Op}_x \circ \mathsf{Op}_y$, and if further $x \circ y \in Q_2^H$, then $H_{x \circ y} = H_x \circ H_y$.

Definition II.2.2. *If $\langle\rangle \in \{x, y\}$, then $x \circ y := x * y$, and if $x = \langle x_1, \ldots, x_k\rangle$ with $x_k = (\alpha, v)$, and $y = \langle y_1, \ldots, y_l\rangle$ with $y_1 = (\beta, w)$, then*

$$x \circ y :\simeq \begin{cases} x * y & : \text{ if } x * y \in Q_2, \\ \langle x_1, \ldots, x_{k-1}, (\beta{+}\alpha, v), y_2, \ldots, y_l\rangle & : \text{ if } v = w, \\ \text{undefined} & : \text{ else.} \end{cases}$$

If x and y are names and $x \circ y$ is defined, then $x \circ y$ is a name, too. Also observe that $\circ$ is associative. Further note, that the reading of $(0, v)$ as an abbreviation for q_0 helps to avoid case distinctions: in the sequel, we often use that $(\alpha{+}1, v)$ is $(1, v) \circ (\alpha, v)$, which thus also holds for $\alpha = 0$.

The following property is essentially trivial, yet important enough to phrase it as a lemma.

Lemma II.2.3. *If $x \circ y \in Q_2$, then $\deg(x \circ y) = \deg(x)$ and $o(x \circ y) \in \mathsf{Lim}(\Omega)$ iff $o(x) \in \mathsf{Lim}(\Omega)$.*

Definition II.2.4. *We say that $x \circ y$ is an expression in normal form, if $\mathsf{lh}(x) = 1$, and either*

(i) $x = (1, v)$, *or*

(ii) $x = (\gamma, v)$ *and* $x \circ y = x * y$.

We write $z =_{NF} x \circ y$ if $z = x \circ y$ and $x \circ y$ is an expression in normal form. Further, if $x \circ y$ is an expression in normal form, then we call x simple.

So if $x \in Q_2$, and $z =_{NF} x \circ y$, then $x = (1, v)$ or $x = (\gamma, v)$, where $v \in Q_1 \cup P_1$ and thus is either q_0, or of the form $(1{+}\beta, q_0)$, (γ, q_0) or $(\gamma, q_0)^-$. Also note that a simple name of degree two is of the form $(1, (1{+}\beta, q_0))$. Each name $x \in Q_2^*$ can be uniquely written in normal form.

Lemma II.2.5. *If $x \in Q_2$, then there are unique $y \in Q_2^*$ and $z \in Q_2$, so that $x =_{NF} y \circ z$.*

Proof If $\deg(x) = 1$, then either $(x)_0 = (\alpha{+}1, q_0)$, $(x)_0 = (1{+}\beta, (\gamma, q_0)^-)$, or $(x)_0 = (\gamma, v)$ with $v \in Q_1 \cup P_1$. Therefore, if $x = (x)_0 * x'$, then either $x = q_1 \circ z$ for $z := (\alpha, q_0) * x'$, or $x = (1, (\gamma, q_0)^-) \circ z$ for $z := (\beta, (\gamma, q_0)^-) * x'$, or $x = (\gamma, v) * z$ for $z := x'$. Further, these representations are unique. And if $\deg(x) = 2$ and $x = (x)_0 * x'$, then $(x)_0$ is of the form $(\alpha{+}1, y')$, and $x =_{NF} (1, y') \circ z$. Again, y' and z are uniquely determined. □

We use this unique normal form to assign approximations $x[\alpha]$ to each $x \in Q_2^*$. Clauses (i)–(iii) address names of degree one, clauses (iv)–(v) names of degree two.

Definition II.2.6. *Let $x =_{NF} y \circ z \in Q_2^*$. Then, $x[\alpha] := y[\alpha] \circ z$, where*

(i) $(1, q_0)[\alpha] = q_0$.

(ii) $(\gamma, v)[\alpha] := (1{+}\alpha, v)$ *if* $\alpha \leq \gamma$, *and else* $(\gamma, v)[\alpha] := (\gamma, v)$.

(iii) $(1, (\gamma, q_0)^-)[\alpha] := (1, (\gamma, q_0)[\alpha])$.

(iv) $(1, (\beta{+}1, q_0))[\alpha] := (1{+}\alpha, (\beta, q_0))$.

(v) $(1, (\gamma, q_0))[\alpha] := (1, (\gamma, q_0)[\alpha])$.

It is immediate by this definition, that $x[\alpha] \leq_2 x$, and if $\alpha < o(x)$, then $x[\alpha] <_2 x$. Actually, if $o(x) = \gamma$, then only approximations $x[\alpha]$ for $\alpha < \gamma$ will matter in the sequel. The case distinction in clause (ii) just assures that $x[\alpha]$ is always defined and in Q_2 (setting $(\gamma, v)[\alpha] := (1{+}\alpha, v)$ for all α would mean that e.g. for $x := (2, (\omega, q_0)) =_{NF} (1, (\omega, q_0)) \circ (1, (\omega, q_0))$, $x[\omega{+}1] \simeq (1, (\omega{+}1, q_0)) \circ (1, (\omega, q_0))$ is not defined).

Definition II.2.7. *If $v \in Q_2 \cup P_1$, then we denote by $v{+}1$ its successor w.r.t. the ordering $(Q_2 \cup P_1, <_2)$ (cf. Definition II.1.9): if $x \in Q_2$, then $x{+}1 := q_1 \circ x$, and if $x^- \in P_1$, then $x^-{+}1 := x$.*

Note that $(\alpha, q_0){+}1 = (\alpha{+}1, q_0)$. Further, $\langle x_1, \ldots, x_k\rangle{+}1 = \langle x_1{+}1, \ldots, x_k\rangle$.

The following properties of $x[\alpha]$ are immediate by the definition. Recall that v ranges over $Q_2 \cup P_1$, while x, y range over Q_2.

Lemma II.2.8. *We have the following.*

(i) $(x{+}1)[\alpha] = x$ *and* $(1, x{+}1)[\alpha] = (1{+}\alpha, x)$.

(ii) *if* $x = (x)_0 * y$, *then* $x[\alpha] := (x)_0[\alpha] * y$,

(iii) $(\beta{+}1, v)[\alpha] := (1, v)[\alpha] \circ (\beta, v)$,

(iv) $(1, v)[\alpha] := (1, v[\alpha])$ *if* $v \neq y{+}1$ *(for some* $y \in Q_2$*).*

We also record the following simple fact, as it is used later in some proofs.

Lemma II.2.9. *Assume that $x \in Q_2^*$ and $x \circ y \in Q_2^*$, and let*

$$\delta_0 := \begin{cases} \delta : & x = (\gamma, v) \wedge (y)_0 = (\delta, v), \\ 0 : & \textit{otherwise.} \end{cases}$$

Then, for each α, $x[\alpha] \circ y = (x \circ y)[\delta_0{+}\alpha]$, and $o(x \circ y) = \delta_0{+}o(x)$. In particular, if $\deg(x) = 2$, then $x[\alpha] \circ y = (x \circ y)[\alpha]$.

Proof The claim obviously holds if $x = (\gamma, v)$. If $x = (\beta{+}1, v)$, then we have two cases: $x \circ y = x * y$, hence $(x * y)[\alpha] = x[\alpha] * y$, or $(y)_0 = (\delta, v)$ and $x \circ y = (1, v) \circ (\delta{+}\beta, v) * z$, where $y = (y)_0 * z$. If $\mathsf{lh}(x) > 1$, then $x = (x)_0 * z$ and $(x \circ y) = (x)_0 * (z \circ y)$. Hence, $(x \circ y)[\alpha] = x_0[\alpha] * (z \circ y) = (x_0[\alpha] * z) \circ y = (x \circ y)[\alpha]$. The second claim is also readily checked. $\square$

The next Lemma tells us in what cases we have that $x = \mathsf{sup}_{<_2}\{x[\alpha] <_2 x : \alpha \in \Omega\}$.

Lemma II.2.10. *If $\deg(x) = 1$ and $o(x) = \gamma$, and $\deg(y) = 2$ and $o(y) = 1$, then*

$$x = \mathsf{sup}_{<_2}\{x[\alpha] : \alpha < \gamma\} \quad \textit{and} \quad y = \mathsf{sup}_{<_2}\{y[\alpha] : \alpha < \Omega\}.$$

Proof By the above Lemma, if $x =_{NF} y \circ z$, then $x[\alpha] = y[\alpha] \circ z$, and $\mathsf{sup}_{\alpha \in I}(y \circ z)[\alpha] = (\mathsf{sup}_{\alpha \in I} y[\alpha]) \circ z$, readily follows. Thus, it suffices to show the claim for simple names. So assume that x and y are as assumed in the lemma, but simple. If x is of the form (γ, z), then the claim is readily observed, and if x is of the form $(1, z^-)$ with $z = (\gamma, q_0)$, then $x' <_2 (1, z^-)$ entails that $x' <_2 x''$, where $x'' = \langle(\beta_1, v_1'), \ldots, (\beta_i, v_i')\rangle <_1 (1, z^-)$ and $v_1' <_1 \ldots < v_i' <_1 z^-$. Hence, there is an α so that $x'' <_2 (1, (1{+}\alpha, q_0)) = (1, z^-)[\alpha] = x[\alpha]$. The first claim follows. As y is of the form $(1, (\beta{+}1, q_0))$, it is immediate that if $z <_2 y$, then already $z <_2 (1{+}\alpha, (\beta, q_0))$ for some α. This yields the second claim. $\square$

If $\deg(x) = 2$ with $o(x) = \gamma$, then $x =_{NF} (1, (\gamma, q_0)) \circ z$. Then, for $y := (1, (\gamma, q_0)^-) \circ z$, we have $x[\alpha] = y[\alpha]$. By the above lemma, $y = \mathsf{sup}_{\alpha<\gamma} y[\alpha] = \mathsf{sup}_{\alpha<\gamma} x[\alpha] <_2 x$. This is where the approximation $x(\alpha) := (1{+}\alpha, (\gamma, q_0)^-) \circ z$ takes over.

The approximation $x(\alpha)$ is only defined for names x with $\deg(x) = 2$. It is defined so that if $o(x) \in \mathsf{Lim}(\Omega)$, then $\mathsf{sup}_\alpha x[\alpha] = x(0)$ and $\mathsf{sup}_\alpha x(\alpha) = x$. Further, it is arranged that $\deg(x(\alpha)) = 1$.

Definition II.2.11. *If $\deg(x) = 2$ and x is simple, then*

(i) $(1, (\gamma, q_0))(\alpha) := ((1{+}\alpha, (\gamma, q_0)^-)$, *and*

(ii) *if $o(x) = 1$, then $x(\alpha) := x[\alpha]{+}1$.*

If $\deg(x) = 2$ *and* $x =_{NF} y \circ z \in Q_2^*$, *then* $x(\alpha) := y(\alpha) \circ z$.

The next lemma lists some easy consequences of this definitions.

Lemma II.2.12. *If* $\deg(x \circ y) = 2$, *then* $x(\alpha) \circ y = (x \circ y)(\alpha)$, *and if* $\deg(z) = 2$ *and* $o(z) = 1$, *then* $z(\alpha) = z[\alpha]{+}1$.

Below, we summarize the properties of the approximations $x[\alpha]$ and $x(\alpha)$ for names of degree two. The proofs are simple and along the line of the proof of Lemma II.2.10, and therefore omitted.

Lemma II.2.13. *If* $x \in Q_2$ *with* $\deg(x) = 2$ *and* $o(x) = \gamma$, *and* $q_0 \neq y \in Q_2^H$ *is not a successor, then*

(i) $x(0) = \mathsf{sup}_{<_2}\{x[\alpha] : \alpha < \gamma\}$, *and* $x = \mathsf{sup}_{<_2}\{x(\alpha) : \alpha < \Omega\}$,

(ii) $y = \mathsf{sup}_{<_2^H}\{y[\alpha] <_2^H y : \alpha \in \Omega\}$.

The following straightforward observation will prove very useful once we have proper definitions of the theories $(\mathsf{T}_x := \mathsf{Op}_x(\mathsf{ACA}_0) : x \in Q_2^*)$, to see that e.g. for names of degree two, $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}_x \to \breve{\mathsf{T}}_{x(0)}$.

Lemma II.2.14. *Assume that* x *is a simple name with* $\deg(x) = 2$. *Then,*

(i) If $o(x) \in \mathsf{Lim}(\Omega)$, *then* $\deg(x(\alpha)) = 1$, $o(x(0)) = o(x)$, $x(0)[\alpha] = x[\alpha]$, $o(x(\gamma)) = \gamma$ *and* $x(\gamma)[\alpha] = x(\alpha)$ *for* $\alpha < \gamma$. *Further,* $x(0) \circ x(\alpha) = x(\alpha{+}1)$.

(ii) If $o(x) = 1$, *then* $\deg(x[\gamma]) = 1$, $o(x[\gamma]) = \gamma$ *and* $x[\gamma][\alpha] = x[\alpha]$ *for* $\alpha < \gamma$. *Further,* $x[0] \circ x[\alpha] = x[\alpha{+}1]$.

We conclude this section by introducing the relation $\rightsquigarrow$ on Q_2 whose transitive closure $\rightsquigarrow^*$ will take the role of the relation $\prec$ in the Representation Theorem when we define internal representations of the operations $(\mathsf{Op}_x : x \in Q_2^*)$, and by defining the map $\cdot^H : Q_2 \to Q_2^H$, which adjust the skewed correspondence between Op_x and H_x and is tailored so that Op_x corresponds to H_{x^H}.

Definition II.2.15. *All all* $x, y \in Q_2$, $y \rightsquigarrow x :\Leftrightarrow (\exists \alpha < o(x))(y = x[\alpha])$. *Further,* $\rightsquigarrow^*$ *is the transitive closure of* $\rightsquigarrow$, *and* $\rightsquigarrow_r^*$ *is the reflexive closure of* $\rightsquigarrow^*$.

Note that e.g. $q_0 \rightsquigarrow q_1$, $q_1 \rightsquigarrow q_2$ and $q_1 \rightsquigarrow q_1^2$, and that further, q_2 and q_1^2 are incomparable w.r.t. $\rightsquigarrow^*$. Further relevant properties of $\rightsquigarrow^*$ are collected below.

Lemma II.2.16.

(i) $(Q_2, \rightsquigarrow^*)$ *is well-founded.*

(ii) $q_0 \rightsquigarrow_r^* x$.

(iii) *If* $y \rightsquigarrow^* x$, *then either* $y \rightsquigarrow x$ *or* $y \rightsquigarrow^* x[\alpha]$ *for some* $\alpha < o(x)$.

(iv) *If* $y \circ z \in Q_2$ *and* $x \circ z \in Q_2$, *then* $y \rightsquigarrow^* x \Rightarrow y \circ z \rightsquigarrow^* x \circ z$.

(v) *If* $1 \leq \alpha < \beta$, *then* $(\alpha, v) \rightsquigarrow^* (\beta, v)$.

(vi) *If* $y \rightsquigarrow^* x$, *then* $(1, y) \rightsquigarrow^* (1, x)$.

(vii) *If* $\alpha < \beta < o(x) = \gamma$, *then* $x[\alpha] \rightsquigarrow^* x[\beta]$.

(viii) *If* $y \rightsquigarrow^* x$ *and* $z \rightsquigarrow^* x$, *then* $y \rightsquigarrow^* z \vee y = z \vee z \rightsquigarrow^* y$.

(ix) *If* $\deg(x) = 2$ *and* $\alpha < \beta$, *then* $x(\alpha) \rightsquigarrow x(\beta)$.

(x) $\mathsf{Wo}_{\rightsquigarrow^*}(x)$.

Proof (i) If $y \rightsquigarrow x$, then $y <_2 x$, therefore, as $(Q_2, <_2)$ is well-founded, also $(Q, \rightsquigarrow^*)$. (ii) Immediate by induction on x along $\rightsquigarrow^*$, using that for $q_0 \neq x$, $x[0] \rightsquigarrow x$. (iii) Directly by definition of $\rightsquigarrow^*$. (iv) By induction on x along $\rightsquigarrow^*$: if $y \rightsquigarrow^* x$, then either $y \rightsquigarrow x$, that is $y = x[\alpha]$ for some $\alpha < o(x)$, and then for some δ_0 (cf. Lemma II.2.9), $x[\alpha] \circ z = (x \circ z)[\delta_0{+}\alpha] \rightsquigarrow x \circ z$, or $y \rightsquigarrow^* x[\alpha] \rightsquigarrow x$ for some $\alpha < o(x) = o(x \circ z)$, hence $y \circ z \rightsquigarrow_{IH}^* x[\alpha] \circ z$, and as above, $x[\alpha] \circ z \rightsquigarrow x \circ z$. (v) By induction on $\beta > \alpha$. If $\beta = \gamma$ is a limit, then $(\alpha, v) \rightsquigarrow (\gamma, v)$ by definition. For the successor case, note that, using (iv), $(\beta, v) \rightsquigarrow_r^* (1, v)[0] * (\beta, v) = (\beta{+}1, v)[0] \rightsquigarrow^* (\beta{+}1, v)$. (vi) By induction on x along $\rightsquigarrow^*$: if $y \rightsquigarrow x$, then $y = x[\alpha]$ for some $\alpha < o(x)$, and if $x = y{+}1$, then $\alpha = 0$ and $(1, y) = (1, y{+}1)[0] \rightsquigarrow (1, x)$, and if $x \neq y{+}1$, then $(1, y) = (1, x[\alpha]) = (1, x)[\alpha] \rightsquigarrow (1, x)$; and if $y \rightsquigarrow^* x[\alpha]$, then $(1, y) \rightsquigarrow_{IH}^* (1, x[\alpha])$, and the claim follows as above. (vii) By (iv), it suffices to consider simple names. Then $x[\alpha]$ is either of the form $(1{+}\alpha, y)$ and the claims is by (v), or $x[\alpha]$ is of the form $(1, (1{+}\alpha, q_0))$, and the claim is by (iv). (viii) If $y \rightsquigarrow^* x$ and $z \rightsquigarrow^* x$, then there are $\alpha, \beta < o(x)$ so that $y \rightsquigarrow_r^* x[\alpha]$ and $z \rightsquigarrow_r^* x[\beta]$. By (v) $y \rightsquigarrow_r^* z$ or $z \rightsquigarrow_r^* y$. (ix) Let $\alpha < \beta$. By (iv), it suffices to consider simple names, so either $(1, y{+}1)(\alpha) = q_1 \circ (1, y{+}1)[\alpha] \rightsquigarrow_{(vi),(iv)}^* (1, y{+}1)[0] \circ (1, y{+}1)[\alpha] =_{L.II.2.14(ii)} (1, y{+}1)[\alpha{+}1] \rightsquigarrow_r^* (1, y{+}1)[\beta] \rightsquigarrow^* q_1 \circ (1, y{+}1)[\beta] = (1, y{+}1)(\beta)$, or $(1, (\gamma, q_0))(\alpha) = (1{+}\alpha, (\gamma, q_0)^-) \rightsquigarrow^* (1{+}\beta, (\gamma, q_0)^-) = (1, (\gamma, q_0))(\beta)$. (x) By (i) and (viii). $\square$

Finally, we define x^H. As already mentioned, we have that e.g. $\mathsf{Op}_{(\alpha,q_0)}\ (\mathsf{p}_1^\alpha)$ corresponds to $H_{(\alpha,q_0)}\ (\mathsf{it}_1^\alpha)$, and that $\mathsf{Op}_{(1,(n,q_0))}\ (\mathsf{p}_2^n\mathsf{p}_1)$ corresponds to $H_{(1,(n,q_0))}\ (\mathsf{It}_2^n(\mathsf{it}))$. As discussed in the first section of this chapter, $\mathsf{Op}_{(1,(\omega,q_0)^-)}$ corresponds to $H_{(1,(\omega,q_0))}$, $\mathsf{Op}_{(1,(\omega,q_0))}$ corresponds to $H_{(1,(\omega+1,q_0))}$, and as we will see $\mathsf{Op}_{(1,(\omega+n,q_0))}$ corresponds to $H_{(1,(\omega+n+1,q_0))}$.

We word the definition of $\cdot^H$ so that is directly extends to the general case. Therefore, we make use of a "correction" $\mathsf{corr}(x) \in \{0,1\}$. For the case $N_0 = 2$, we need $\mathsf{corr}(x)$ only for $x \in Q_1$, and if $x \in Q_1$, then $\mathsf{corr}(x) = 1$ iff $(\omega, q_0) \leq x$, that is, if x is of the form $(\gamma{+}n, q_0)$. Recall that all names in Q_1^* have length one. However, with regard to the general case, we define corr for each $x \in Q_2$. The idea is that $\mathsf{corr}(x) = 1$ if $x = (\gamma, y) * z$, and it is taken care that $\mathsf{corr}(x{+}1) = \mathsf{corr}(x){+}1$.

Definition II.2.17. *We define* $\mathsf{corr} : Q_2 \to \{0,1\}$ *and* $\cdot^H : Q_2 \to Q_2^H$ *as follows.*

(i) $\mathsf{corr}(x) := 1$ *if* $\exists y, n[x = y{+}n \wedge \deg(y) = 1 \wedge o(y) \in \mathsf{Lim}(\Omega)]$*; else* $\mathsf{corr}(x) := 0$.

(ii) $q_0^H := q_0$, $(\alpha, y^-)^H := (\alpha, y^H)$ *and* $(\alpha, y)^H := (\alpha, y^H{+}\mathsf{corr}(y))$,

(iii) if $k > 1$, *then* $\langle x_1, \ldots, x_k\rangle^H := \langle x_1^H, \ldots, x_k^H\rangle$.

It is easily checked that indeed $\cdot^H : Q_2 \to Q_2^H$. Also note that $(\alpha, q_0)^H = (\alpha, q_0)$, $(\alpha, (\gamma, q_0)^-)^H := (\alpha, (\gamma, q_0))$ and $(\alpha, (\gamma{+}n, q_0))^H := (\alpha, (\gamma{+}n{+}1, q_0))$. Moreover, if $\deg(x) = 2$, then $o(x^H) = 1$ and by Lemma II.2.10, $x = \mathsf{sup}_\alpha x^H[\alpha]$.

Also the following is an easy consequence of the above definition.

Lemma II.2.18. *If* $x, y \in Q_2^*$, *then* $(x \circ y)^H = x^H \circ y^H$.

We close by a technical lemma, which will allows us show that if $\deg(x) = 1$ and $o(x) = \gamma$, then $g_{x^H[\alpha]} \trianglelefteq g_{(x[\alpha])^H} \trianglelefteq g_{(x^H[\alpha+1])}$, and if $\deg(x) = 2$, then $g_{x^H[\alpha]} \trianglelefteq g_{(x(\alpha))^H} \trianglelefteq g_{x^H[\alpha+1]}$.

Lemma II.2.19. *Let* $x \in Q_2$.

(i) If $\deg(x) = 1$ *and* $o(x) \in \mathsf{Lim}(\Omega)$, *then* $x^H[\alpha] \rightsquigarrow_r^* (x[\alpha])^H \rightsquigarrow_r^* x^H[\alpha{+}1]$.

(ii) If $\deg(x) = 2$, *then* $x^H[\alpha] \rightsquigarrow_r^* (x(\alpha))^H \rightsquigarrow_r^* x^H[\alpha{+}1]$.

Proof Let $x =_{NF} y \circ z$. Then $x^H = y^H \circ z^H$. Hence $x^H[\alpha] = y^H[\alpha] \circ z^H$, $(x[\alpha])^H = (y[\alpha])^H \circ z^H$ and $(x(\alpha))^H = (y(\alpha))^H \circ z^H$. By Lemma II.2.16 (iv), it thus suffices to check the claims for simple names. We just check the first claim of (ii), i.e. that $x^H[\alpha] = (x(\alpha))^H$ or $x^H[\alpha]{+}1 = (x(\alpha))^H$. The rest is verified similarly. For $n < \omega$ and limits γ,

$$\begin{array}{ccc}
(1,(n{+}1,q_0)) & \xrightarrow{\cdot^H} & (1,(n{+}1,q_0)) \\
\Big\downarrow{\scriptstyle \cdot(\alpha)} & & \Big\downarrow{\scriptstyle \cdot[\alpha]+1} \\
(1{+}\alpha,(n,q_0)){+}1 & \xrightarrow{\cdot^H} & (1{+}\alpha,(n,q_0)){+}1
\end{array}
\qquad
\begin{array}{ccc}
(1,(\gamma,q_0)) & \xrightarrow{\cdot^H} & (1,(\gamma{+}1,q_0)) \\
\Big\downarrow{\scriptstyle \cdot(\alpha)} & & \Big\downarrow{\scriptstyle \cdot[\alpha]} \\
(1{+}\alpha,(\gamma,q_0)^-) & \xrightarrow{\cdot^H} & (1{+}\alpha,(\gamma,q_0))
\end{array}$$

And for $\beta > \omega$,

$$\begin{tikzcd}
(1,(\beta{+}1,q_0)) \arrow[r, ".H"] \arrow[d, "\cdot(\alpha)"'] & (1,(\beta{+}2,q_0)) \arrow[d, "\cdot[\alpha]+1"'] \\
(1{+}\alpha,(\beta,q_0)){+}1 \arrow[r, ".H"] & (1{+}\alpha,(\beta{+}1,q_0)){+}1
\end{tikzcd}$$

□

II.3 Properties of the functionals $(H_x : x \in Q_2^H)$

In this short section, we collect some basic properties of our functionals. Below, we write also H_x^{+0} for H_x and H_x^{+1} for H_x^+.

The next claim states an expected property of $\circ$ and is an immediate consequence of its definition and Definition II.1.5.

Lemma II.3.1. *For $n \in \{0,1\}$, if $x \circ y \in Q_{n+1}^H$, then $H_{x\circ y}^{+(1-n)} = H_x^{+(1-n)} \circ H_y^{+(1-n)}$.*

We have already seen that for $q_0 \neq x \in Q_2^H$, $H_x \in \Omega^{(1)}$ and $H_x \subseteq \mathsf{it}$ and for $q_0 \neq y \in Q_1^H$, $H_x^+ \in \Omega^{(2)}$, and $H_y \subseteq \mathsf{It}$. More generally, we have the following.

Lemma II.3.2. *If $y \rightsquigarrow_r^* x \in Q_1^H$, then $H_x^+ \subseteq H_y^+$, and if $y \rightsquigarrow_r^* x \in Q_2^H$, then $H_x \subseteq H_y$.*

Proof Since $\mathsf{It} \in \Omega^{(2)}$, we have for each $\beta \leq \alpha$, $\mathsf{It}^\alpha \subseteq \mathsf{It}^\beta$, which yields (i), since $x \in Q_1^H$ means that $x = (\alpha, q_0)$, and if further $y \rightsquigarrow^* (\alpha, q_0)$, then $y = (\beta, q_0)$ for some $\beta < \alpha$.
The second claim is by induction along $\rightsquigarrow^*$. If $x = y$, the claim holds trivially, hence assume that $y \rightsquigarrow^* x$. We do a case distinction on the form of x.

(i) $y \rightsquigarrow^* x =_{NF} (1, x') \circ z$. If $x' = q_0$, then $y \rightsquigarrow_r^* z$ and by I.H. $H_z \subseteq H_y$. Thus, $H_x = \mathsf{it} \circ H_z \subseteq H_z \subseteq H_y$. If $x' = q_1$, then $y \rightsquigarrow_r^* (1, q_0) * z = x[0]$. Using the I.H. and that $\mathsf{It}(\mathsf{it}) \subseteq \mathsf{it}$, $H_x \subseteq H_{x[0]} \subseteq H_y$. If $x' \neq q_0$ and $x' \neq q_1$, then there is an $\alpha < o(x)$, so that $y \rightsquigarrow_r^* x[\alpha] = (1, x'[\alpha]) * z$. By (i), $H_{x'}^+ \subseteq H_{x'[\alpha]}^+$, and so $H_x = H_{x'}^+(\mathsf{it}) \circ H_z \subseteq H_{x'[\alpha]}^+(\mathsf{it}) \circ H_z = H_{x[\alpha]} \subseteq_{IH} H_y$.

(ii) $y \rightsquigarrow^* x =_{NF} (\gamma, x') * z$. Then there is an $\alpha < o(x)$, so that $y \rightsquigarrow_r^* x[\alpha] = (1{+}\alpha, x') \circ z$. Since $H_{x'}^+(\mathsf{it}) \in \Omega^{(1)}$, $(H_{x'}^+(\mathsf{it}))^\gamma \subseteq (H_{x'}^+(\mathsf{it}))^{1+\alpha}$, and so $H_x = (H_{x'}^+(\mathsf{it}))^\gamma \circ H_z \subseteq (H_{x'}^+(\mathsf{it}))^{1+\alpha} \circ H_z \subseteq_{IH} H_y$.

□

Next, we let $f \in \Omega^{(0)}$, and collect some properties of the functions $(f_x : x \in Q_2^H)$. Recall that $f_x := H_x(f)$, f is identified with its range and that $f' = \{\alpha : f(\alpha) = \alpha\}$.

Lemma II.3.3. *If* $\deg(x) = 2$ *and* $o(x) = 1$*, then* $f_x(\alpha) = f_{x[1+\alpha]}(0)$*.*

Proof If $\deg(x) = 2$ and $o(x) = 1$, then x is of the form $x =_{NF} (1, (\beta{+}1, q_0)) \circ z$, and $f_x(\alpha) = H_x[f, \alpha] = (\mathsf{It}^{\beta+1}(\mathsf{it}) \circ H_z)[f, \alpha] = \mathsf{It}^{\beta+1}[\mathsf{it}, f_z, \alpha] = (\mathsf{It}^{\beta}(\mathsf{it}))^{2+\alpha}[f_z, 0] = f_{(2+\alpha,(\beta,q_0))\circ z}(0) = f_{x[1+\alpha]}(0)$. □

Lemma II.3.4. *For each* $x \in Q_2^H$ *with* $o(x) = \gamma$*, we have*

(i) if $\xi < \gamma$*, then* $f_{x[\xi+1]} \subseteq f_{x[\xi]+1}$*,*

(ii) if $\xi < \gamma$*, then* $f_{x[\xi+2]} \subseteq f'_{x[\xi]}$*,*

(iii) $f_x = \bigcap_{\xi<\gamma} f_{x[\xi]} = \bigcap_{\xi<\gamma} f'_{x[\xi]}$*.*

Proof (i) $x =_{NF} y \circ z$ for either $y = (\gamma, (\beta, q_0))$, and then $H_{y[\xi+1]} = \mathsf{It}^{\beta}(\mathsf{it}) \circ H_{y[\xi]} \subseteq \mathsf{it} \circ H_{y[\xi]} = H_{y[\xi]+1}$, since $\mathsf{It}^{\beta}(\mathsf{it}) \subseteq \mathsf{it}$ by Lemma II.1.6, or $y = (1, (\gamma, q_0))$, and $H_{y[\xi+1]} = \mathsf{It} \circ H_{y[\xi]} \subseteq \mathsf{it} \circ H_{y[\xi]} = H_{y[\xi]+1}$ by Lemma II.1.7. The claim the follows. (ii) By (i), and since $\mathsf{it} \subseteq \mathsf{sh}$ (cf. Lemma I.3.15), we have $f_{x[\xi+2]} \subseteq \mathsf{it}(f_{x[\xi+1]}) \subseteq \mathsf{sh}(f_{x[\xi+1]})$. And as sh is monotone by Lemma I.3.17, $\mathsf{sh}(f_{x[\xi+1]}) \subseteq (\mathsf{sh} \circ \mathsf{it})(f_{x[\xi]}) = f'_{x[\xi]}$. (iii) The first equality is by definition of iteration of functionals, and the second follows using (ii) and that $f' \subseteq f$, and therefore $f_{x[\alpha+2]} \subseteq f'_{x[\alpha]} \subseteq f_{x[\alpha]}$. □

Lemma II.3.5. *For each* $x \in Q_2^H$ *and each* $y \in Q_2^H$ *with* $o(y) = \delta_0{+}\gamma$*, we have*

(i) $f_x(\gamma) = \mathsf{sup}\{f_x(\delta_0{+}\xi) : \xi < \gamma\}$*,*

(ii) $f_y(0) = \mathsf{sup}\{s_0(\xi) : \xi < \gamma\}$ *and* $f_y(\alpha{+}1) = \mathsf{sup}\{s_{f_y(\alpha)+1}(\xi) : \xi < \gamma\}$*, where* $s_\beta(0) = \beta{+}1$*,* $s_\beta(\xi{+}1) := f_{y[\delta_0+\xi]}(s_\beta(\xi))$ *and* $s_\beta(\gamma') := \mathsf{sup}_{\xi<\gamma'} s_\beta(\xi)$*.*

Proof (i) By Lemma II.1.6, $f_x \in \Omega^{(0)}$ and thus normal. (ii) By Lemma II.3.4 (iii), we have that $f_{y[\delta_0+\xi+2]} \subseteq f'_{y[\delta_0+\xi]}$ and $f_y = \bigcap_{\xi<\gamma} f_{y[\delta_0+\xi]}$. Now the claim is due to Lemma I.3.19. □

II.4 The operations $(\mathsf{Op}_x : x \in Q_2^*)$

In this section, we first provide the proper definition of the operations $(\mathsf{Op}_x : x \in Q_2^*)$ and $(\mathsf{Op}_x^+ : x \in Q_1^*)$. This allows us to prove a collection of properties of these operations in ACA_0 required for the modular ordinal analysis in the next section. We like to highlight the following two: (i) is a useful generalization of Lemma I.2.14, and (ii) singles out an essetial step of the modular ordinal analysis.

(i) for each open Π^1_2-sentence $\check{\mathsf{T}}'$, $\mathsf{T}^\epsilon \vdash y \in Q_1^* \wedge \mathsf{Op}_{(1,y)}(\check{\mathsf{T}}) \wedge \check{\mathsf{T}}' \rightarrow \mathsf{Op}_{(1,y)}(\check{\mathsf{T}}')$, (cf. Corollary II.4.7).

(ii) $\mathsf{T}^\epsilon \vdash \deg(x) = 2 \wedge \breve{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\{\alpha : \breve{\mathsf{T}}_{x(\alpha)}\})$ (cf. Lemma II.4.9).

Then, we show that the proper Definition II.4.1 of Op_x and the provisional Definition II.1.12 indeed agree.

From now on, it is assumed that we have primitive recursive relations that are formalized versions of $\alpha < \beta$, $(Q_2, <_2)$, $\rightsquigarrow$, $\rightsquigarrow^*$, and primitive recursive functions formalizing $\deg(x)$, $\circ$ and $\cdot^H$. To emphasis that we now work within a formal theory, we write $\alpha \lhd \beta$ for $\alpha < \beta$. The other function- and relationsymbols are overloaded. It is further assumed that we have a recursive function, provably total in ACA_0, that computes the ordinal notation $v(x, \alpha)$ of $g_x(\alpha)$ from a code of the ordinal α and a code of the name x. How to find notations for the ordinals used in this chapter, and how to compute the ordinal notation $v(x, \alpha)$ of $g_x(\alpha)$, is detailed in Chapter IV.

II.4.1 The proper definition of $(\mathsf{Op}_x : x \in Q_2^*)$

To employ the Representation Theorem I.2.26, we have to coach the definition of the operation Op_x into the following form:

$$\mathsf{Op}_x(\breve{\mathsf{T}}) \leftrightarrow (\forall y \rightsquigarrow x)(\mathsf{Op}^\vartheta_{f(y,x)}(\widehat{\mathsf{Op}}_y(\breve{\mathsf{T}}))).$$

Recall that $\widehat{\mathsf{Op}}^\vartheta_u(\breve{\mathsf{T}}) := (0 = u \wedge \breve{\mathsf{T}}) \vee (0 \neq u \wedge \mathsf{Op}^\vartheta_u(\breve{\mathsf{T}}))$, an abbreviation that we use since we cannot represent directly the identity operation by an $\mathsf{L}_2(\mathsf{P})$-formula (cf. page 26). Further, $y \rightsquigarrow x$ iff $(\exists \alpha \lhd o(x))(y = x[\alpha])$. Since we have for names of the form $x := (1, (\beta{+}1, q_0))$ that $o(x) = 1$, only $x[0] = (1, (\beta, q_0)) \rightsquigarrow x$. And for names of the form $z := (1, (\gamma, q_0))$ where $o(z) = \gamma$, we have that $z[\alpha] = (1, (1{+}\alpha, q_0)) \rightsquigarrow z$ for each $\alpha \lhd \gamma$.

Next observe, that by the provisional definition of the operations Op_x, (Definition II.1.12), we have the following.

(i) $\mathsf{Op}_{x+1} \Leftrightarrow \mathsf{p}_1\mathsf{Op}_x$ and $\mathsf{Op}^+_{(1+\alpha,q_0)} \Leftrightarrow \mathsf{p}_2^{1+\alpha}$,

(ii) $\mathsf{Op}_{(1,(\beta+1,q_0))} \Leftrightarrow \mathsf{p}_2^{\beta+1}\mathsf{p}_1 \Leftrightarrow \mathsf{p}_2 \circ \mathsf{Op}_{(1,(\beta,q_0))}$,

(iii) $\mathsf{Op}_{(1,(\gamma,q_0))} \Leftrightarrow \mathsf{p}_2^\gamma\mathsf{p}_1 \Leftrightarrow (\forall \alpha \lhd \gamma)(\mathsf{p}_2 \circ \mathsf{Op}_{(1,(1+\alpha,q_0))})$,

(iv) $\mathsf{Op}_{(1,(\gamma,q_0)^-)} \Leftrightarrow (\forall \alpha \lhd \gamma)(\mathsf{p}_1\mathsf{p}_2^{1+\alpha}\mathsf{p}_1) \Leftrightarrow (\forall \alpha \lhd \gamma)(\mathsf{p}_1 \circ \mathsf{Op}_{(1,(1+\alpha,q_0))})$.

This allows us to read off the ingredients of the proper definition of Op_x and Op^+_x which supersedes the provisional Definition II.1.12 (φ_{p_1} and φ_{p_2} are as fixed in Definition I.2.15).

Definition II.4.1. *Let* $\vartheta(u) := (u = 1 \wedge \varphi_{\mathsf{p}_1}) \vee (u = 2 \wedge \varphi_{\mathsf{p}_2})$, $f(y, x) := \deg(x)$, $f^+(y, x) := \deg(x)+1$, *and*

$$\varphi(u) := \varphi^{f,\rightsquigarrow^*,\rightsquigarrow,\vartheta}(u) \quad \text{and} \quad \varphi^+(u) := \varphi^{f^+,\rightsquigarrow^*\restriction Q_1,\rightsquigarrow\restriction Q_1,\vartheta}(u),$$

where $\varphi^{f^+,\rightsquigarrow^*\restriction Q_1,\rightsquigarrow\restriction Q_1,\vartheta}(u)$ *is the formula claimed to exists in Theorem I.2.26, and defined in the Appendix (see Theorem A.1.2 and Definition A.1.11). Then,*

$$\mathsf{Op}_x(\breve{\mathsf{T}}) := \mathsf{Op}_x^{\varphi}(\breve{\mathsf{T}}) \text{ and } \mathsf{Op}_x^+(\breve{\mathsf{T}}) := \mathsf{Op}_x^{\varphi^+}(\breve{\mathsf{T}}) \text{ and } \breve{\mathsf{T}}_x := \widehat{\mathsf{Op}}_x^{\varphi}(\breve{\mathsf{T}}).$$

Note that $\breve{\mathsf{T}}_x$ iff $(x = 0 \wedge \breve{\mathsf{T}}) \vee (x \neq 0 \wedge \mathsf{Op}_x(\breve{\mathsf{T}}_x))$. The following Lemma is an instance of the Representation Theorem and reviews the properties of Op_x and Op_x^+.

Lemma II.4.2. *The maps* $\breve{\mathsf{T}} \mapsto \mathsf{Op}_x(\breve{\mathsf{T}})$ *and* $\breve{\mathsf{T}} \mapsto \mathsf{Op}_x^+(\breve{\mathsf{T}})$ *are operations that satisfy the following properties (provable in* T^ϵ*). Below,* $m \in \{0, 1\}$.

(i) $\mathsf{Op}_x(\breve{\mathsf{T}}) \to x \in Q_2^* \wedge \mathsf{Wo}_{\rightsquigarrow^*}(x)$, *and* $\mathsf{Op}_x^+(\breve{\mathsf{T}}) \to x \in Q_1^* \wedge \mathsf{Wo}_{\rightsquigarrow^*}(x)$,

(ii) if $q_1 \rightsquigarrow^* x \in Q_1$, *then* $\mathsf{Op}_x^+ \Leftrightarrow (\forall \alpha \lhd o(x))(\mathsf{p}_2 \circ \mathsf{Op}_{x[\alpha]}^+)$.

(iii) if $q_1 \rightsquigarrow^* x \in Q_2$ *and* $\deg(x) = m+1$, *then* $\mathsf{Op}_x \Leftrightarrow (\forall \alpha \lhd o(x))(\mathsf{p}_{m+1} \circ \mathsf{Op}_{x[\alpha]})$.

Further, as $\varphi(u)$ *strongly implies* p_1 *(this is also by the Representation Theorem), we have that* $\mathsf{T}^\epsilon \vdash (\forall x \in Q_2^*)(\mathsf{Op}_x \Rightarrow \mathsf{p}_1)$.

Since $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ is Π^1_{n+2}, also the following is readily observed.

Lemma II.4.3. *If* $x \in Q_1^*$, *then* $\mathsf{Op}_x^+(\breve{\mathsf{T}})$ *is* Π^1_3*; if* $x \in Q_2^*$, *then* $\mathsf{Op}_x(\breve{\mathsf{T}})$ *is* $\Pi^1_{1+\deg(x)}$.

II.4.2 Properties of Op_x and Op_x^+

In this subsection, we prove properties of the operations Op_x in ACA_0. Recall that we write e.g. $\mathsf{Op}_x \Rightarrow \mathsf{Op}_y$, if for all open L_2-sentences $\breve{\mathsf{T}}$, $\mathsf{T}^\epsilon \vdash \mathsf{Op}_x(\breve{\mathsf{T}}) \to \mathsf{Op}_y(\breve{\mathsf{T}})$.

The next couple of lemmas are all proved by induction along $\lhd$ or along $\rightsquigarrow^*$ using Theorem I.4.2 or Corollary I.4.3. For all these proofs, we let $A(x)$ express the claim, and show, working informally in ACA_0, that $(\forall y \rightsquigarrow^* x)a(x) \to A(x)$, or that $B \wedge b \wedge (\forall y \prec x)c(y) \to C(x)$ in case that $A(u) := B \wedge b \to C(u)$. We refer to the assumption $(\forall y \rightsquigarrow^* x)a(x)$ also as I.H. (in the sense of Theorem I.4.2). Since $\mathsf{T}^\epsilon \vdash \neg\breve{\mathsf{T}}_x \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_{\rightsquigarrow^*}(x))$ (and trivially, $\mathsf{Wo}_{\rightsquigarrow^*}(\alpha, v)$ implies $\mathsf{Wo}_{\lhd}(\alpha)$), it is in all cases readily observed that indeed $\mathsf{T}^\epsilon \vdash A(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_{\rightsquigarrow^*}(x))$ (or $\mathsf{T}^\epsilon \vdash A(\alpha) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_{\lhd}(\alpha)))$, so that Theorem I.4.2 applies and allows us to conclude $(\forall y \rightsquigarrow^* x)A(x)$.

Lemma II.4.4. *The following is provable in* T^ϵ*. For all* $x, y \in Q_2^*$,

(i) *if* $0 \lhd \beta \lhd \alpha$, *then* $\mathsf{Op}^{+}_{(\alpha,q_0)} \Rightarrow \mathsf{Op}^{+}_{(\beta,q_0)}$,

(ii) *if* $y \rightsquigarrow^{*} x \in Q^{*}_{2}$, *then* $\breve{\mathsf{T}}_x \to \mathsf{p}_1\breve{\mathsf{T}}_y$,

(iii) *if* $x, y \in Q^{*}_{1}$ *and* $x \circ y \in Q^{*}_{1}$, *then* $\mathsf{Op}^{+}_{x\circ y} \Leftrightarrow \mathsf{Op}^{+}_{x} \circ \mathsf{Op}^{+}_{y}$,

(iv) *if* $x, y \in Q^{*}_{2}$ *and* $x \circ y \in Q^{*}_{2}$, *then* $\mathsf{Op}_{x\circ y} \Leftrightarrow \mathsf{Op}_{x} \circ \mathsf{Op}_{y}$,

(v) *if* $x \in Q^{*}_{1}$, *then* $\mathsf{Op}_{(1,x)} \Leftrightarrow \mathsf{Op}^{+}_{x}\mathsf{p}_1$.

Proof (i) Let $A(\alpha) := 0\lhd\beta\lhd\alpha\wedge\mathsf{Op}^{+}_{(\alpha,q_0)}(\breve{\mathsf{T}}) \to \mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$. If $\alpha\unlhd\beta$, there is nothing to show. So assume that $\beta\lhd\alpha$. If $\alpha = \alpha'+1$, then $\mathsf{Op}^{+}_{(\alpha,q_0)}(\breve{\mathsf{T}})$ iff $\mathsf{p}_2\mathsf{Op}^{+}_{(\alpha',q_0)}(\breve{\mathsf{T}})$. By I.H., we have that for each X, $\mathsf{Op}^{+}_{(\alpha',q_0)}(\breve{\mathsf{T}})\restriction X \to \mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})\restriction X$. Since p_2 is an operation, $\mathsf{p}_2\mathsf{Op}^{+}_{(\alpha',q_0)}(\breve{\mathsf{T}}) \to \mathsf{p}_2\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$ follows. As further, $\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$ is Π^1_3, $\mathsf{p}_2\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$ implies $\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$ by Lemma I.2.12. Hence $A(\alpha)$ follows. And if α is a limit, then $\mathsf{Op}^{+}_{(\alpha,q_0)}(\breve{\mathsf{T}})$ iff $(\forall\xi \lhd \alpha)\mathsf{p}_2\mathsf{Op}^{+}_{(1+\xi,q_0)}(\breve{\mathsf{T}})$, and since $\beta \lhd \alpha$ by assumption, $\mathsf{Op}^{+}_{(\alpha,q_0)}(\breve{\mathsf{T}})$ implies $\mathsf{p}_2\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$, and $\mathsf{Op}^{+}_{(\beta,q_0)}(\breve{\mathsf{T}})$ follows as above.
(ii) Let $A(x) := y \rightsquigarrow^{*} x \in Q^{*}_{2} \wedge \breve{\mathsf{T}}_x \to \mathsf{p}_1\breve{\mathsf{T}}_y$. If $x = y+1$, then $x[0] = y$, and the claim is by definition of $\breve{\mathsf{T}}_x$. Otherwise, there is an $\alpha \lhd o(x)$ so that $y = x[\alpha]$ or $y \rightsquigarrow^{*} x[\alpha]$. In the first case, $\breve{\mathsf{T}}_x \to \mathsf{p}_1\breve{\mathsf{T}}_y$ by definition of $\breve{\mathsf{T}}_x$ and since, in any case, $\mathsf{p}_{\deg(x)} \Rightarrow \mathsf{p}_1$. And if $x[\alpha] \rightsquigarrow^{*} x$, then the I.H. yields for each X, $\breve{\mathsf{T}}_{x[\alpha]}\restriction X \to \mathsf{p}_1\breve{\mathsf{T}}_y\restriction X$. Since p_1 is an operation, we obtain $\mathsf{p}_1\breve{\mathsf{T}}_{x[\alpha]} \to \mathsf{p}_1^2\breve{\mathsf{T}}_y$. As $\breve{\mathsf{T}}_x$ iff $(\forall\alpha \lhd o(x))\mathsf{p}_{\deg(x)}\breve{\mathsf{T}}_{x[\alpha]}$ and $\mathsf{p}_2 \Rightarrow \mathsf{p}_1$, we have also $\breve{\mathsf{T}}_x \to \mathsf{p}_1\breve{\mathsf{T}}_{x[\alpha]}$. By Lemma I.2.12, $\mathsf{p}_1^2\breve{\mathsf{T}}_y \to \mathsf{p}_1\breve{\mathsf{T}}_y$, hence $\breve{\mathsf{T}}_x \to \mathsf{p}_1\breve{\mathsf{T}}_y$ follows.
(iii) As (vi), but simpler.
(iv) Let $A(x) := q_0 \neq x \wedge x \circ y \in Q^{*}_{2} \to [\breve{\mathsf{T}}_{x\circ y} \leftrightarrow \mathsf{Op}_x(\breve{\mathsf{T}}_y)]$. If x or y is q_0, then the claim is trivial, so assume otherwise. Now let m so that $m+1 = \deg(x)$, and δ_0 so that $(\forall\alpha \lhd o(x))((x \circ y)[\delta_0+\alpha] = x[\alpha] \circ y)$, and therefore $o(x \circ y) = \delta_0+o(x)$ (cf. Lemma II.2.9). As $\breve{\mathsf{T}}_{x\circ y}$ iff $(\forall\alpha \lhd o(x \circ y))\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{(x\circ y)[\alpha]})$, we have that $\breve{\mathsf{T}}_{x\circ y}$ implies $(\forall\alpha \lhd o(x))\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{x[\alpha]\circ y})$. For each $\alpha \lhd o(x)$ and each X, the I.H. yields $\mathsf{Op}_{x[\alpha]\circ y}(\breve{\mathsf{T}})\restriction X \leftrightarrow \mathsf{Op}_{x[\alpha]}(\breve{\mathsf{T}}_y)\restriction X$. Since p_{m+1} is an operation, we obtain $(\mathsf{p}_{m+1} \circ \mathsf{Op}_{x[\alpha]\circ y})(\breve{\mathsf{T}})$ iff $(\mathsf{p}_{m+1} \circ \mathsf{Op}_{x[\alpha]})(\breve{\mathsf{T}}_y)$. Now $\breve{\mathsf{T}}_{x\circ y} \to \mathsf{Op}_x(\breve{\mathsf{T}}_y)$ follows. For the converse direction, note that $(\forall\alpha\lhd\gamma)\mathsf{p}_{m+1}\mathsf{Op}_{x[\alpha]}(\breve{\mathsf{T}}_y)$ yields $(\forall\alpha\lhd\gamma)\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{(x\circ y)[\delta_0+\alpha]})$. Using (ii) yields $(\forall\alpha \lhd \delta_0+\gamma)\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{(x\circ y)[\delta_0+\alpha]})$. (v) Similar, using that $\deg(1, x) = \deg(x)+1$, $o(1, x) = o(x)$, and $(1, x)[\alpha] = (1, x[\alpha])$. $\square$

The next Lemma generalizes Lemma I.2.14.

Lemma II.4.5. *For each open* Π^1_2*-sentence* $\breve{\mathsf{T}}'$,

$$\mathsf{T}^{\epsilon} \vdash x \in Q^{*}_{1} \wedge \mathsf{Op}^{+}_{x}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}^{+}_{x}(\breve{\mathsf{T}}').$$

Proof Let $\breve{\mathsf{T}}'$ be an open Π^1_2-sentence, and $A(x) := x \in Q^*_1 \wedge \mathsf{Op}^+_x(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}^+_x(\breve{\mathsf{T}}')$. Trivially, $A(q_0)$, and if $x = q_1$, then by Lemma I.2.14, $\mathsf{p}_2(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ yields $\mathsf{p}_2(\breve{\mathsf{T}}')$. If $x = y{+}1$, then by definition, $\mathsf{Op}^+_{y+1}(\breve{\mathsf{T}})$ iff $\mathsf{p}_2(\mathsf{Op}^+_y(\breve{\mathsf{T}}))$. By Lemma I.2.14, $\mathsf{p}_2(\mathsf{Op}^+_y(\breve{\mathsf{T}})) \wedge \breve{\mathsf{T}}'$ yields $\mathsf{p}_2(\mathsf{Op}^+_y(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}')$. By I.H.,$\forall X[(\mathsf{Op}^+_y(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}')\restriction X \to \mathsf{Op}^+_y(\breve{\mathsf{T}}')\restriction X]$. Since p_2 is an operation, we obtain $\mathsf{p}_2(\mathsf{Op}^+_y(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}') \to \mathsf{p}_2(\mathsf{Op}^+_y(\breve{\mathsf{T}}'))$. Hence, $\mathsf{Op}^+_x(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ implies $\mathsf{Op}^+_x(\breve{\mathsf{T}}')$. The limit case is shown analogously. $\square$

The following observations are essentially trivial, but nonetheless important.

Lemma II.4.6. *Let $(1, y) \in Q_2$ be a simple name of degree two (so $y \in Q^*_1$). Then,*

(i) $\mathsf{Op}^+_y \Rightarrow \mathsf{Op}_{(1,y)}$.

(ii) $\mathsf{Op}_{(1,y)} \Leftrightarrow \mathsf{Op}_{(1,y)}\mathsf{p}_1$.

Proof (i) $\mathsf{Op}^+_y(\breve{\mathsf{T}})$ implies $\mathsf{p}_1(\breve{\mathsf{T}})$ which is Π^1_2, thus by the above lemma, $\mathsf{Op}^+_y(\breve{\mathsf{T}})$ implies $\mathsf{Op}^+_y\mathsf{p}_1(\breve{\mathsf{T}})$, that is $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}})$ by Lemma II.4.4 (v). (ii) $\mathsf{Op}_{(1,y)} \Leftrightarrow \mathsf{Op}^+_y\mathsf{p}_1$, so (i) implies $\mathsf{Op}_{(1,y)}\mathsf{p}_1$. For the converse direction, note that $\mathsf{Op}_{(1,y)}\mathsf{p}_1 \Leftrightarrow \mathsf{Op}^+_y\mathsf{p}^2_1$. As $\mathsf{p}_2 \Rightarrow \mathsf{p}_1$ by Lemma I.2.12, also $\mathsf{Op}^+_y\mathsf{p}^2_1 \Rightarrow \mathsf{Op}^+_y\mathsf{p}_1$, and the claim is by (i). $\square$

This allows us to state the following useful variant of Lemma III.6.6.

Corollary II.4.7. *For each open Π^1_2-sentence $\breve{\mathsf{T}}'$,*

$$\mathsf{T}^\epsilon \vdash y \in Q^*_1 \wedge \mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}').$$

Proof Since $\mathsf{Op}_{(1,y)} \Leftrightarrow \mathsf{Op}^+_y\mathsf{p}_1$, $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ implies $\mathsf{Op}^+_y(\mathsf{p}_1(\breve{\mathsf{T}})) \wedge \breve{\mathsf{T}}'$, which by Lemma II.4.5 yields $\mathsf{Op}^+_y(\breve{\mathsf{T}}')$, which in turn yields $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}')$ by Lemma II.4.6. $\square$

The lemma following the next auxiliary lemma is a key step in our modular ordinal analysis.

Lemma II.4.8. *Assume that x is a simple name with* $\deg(x) = 2$. *Then,*

(i) $\mathsf{Op}_x \Rightarrow \mathsf{Op}_{x(0)}$,

(ii) $\mathsf{Op}_{x(0)} \circ \mathsf{Op}_{x(\alpha)} \Rightarrow \mathsf{Op}_{x(\alpha+1)}$,

(iii) if $o(x) = 1$, then for each limit γ, $\mathsf{Op}_{x[\gamma]} \Leftrightarrow (\forall \alpha \lhd \gamma)\mathsf{Op}_{x(\alpha)}$,

(iv) if $o(x) \in \mathsf{Lim}(\Omega)$, then $\mathsf{Op}_{x(\gamma)} \Leftrightarrow (\forall \alpha \lhd \gamma)\mathsf{Op}_{x(\alpha)}$.

Proof We use that if $\deg(y) = m{+}1$, then Op_x iff $(\forall \alpha \lhd o(y))\mathsf{p}_{m+1}\mathsf{Op}_{x[\alpha]}$ (Lemma II.4.2 (iii)), and that if $y \circ z \in Q^*_2$, then $\mathsf{Op}_{y\circ z} \Leftrightarrow \mathsf{Op}_y \circ \mathsf{Op}_z$ (Lemma II.4.4 (iv)). Now assume that x is simple and $\deg(x) = 2$. By Definition II.2.11, $\deg(x(\alpha)) = 1$, and if $o(x) = 1$ then $x(\alpha) = q_1 \circ x[\alpha]$.

(i) If $o(x) = 1$, then Op_x iff $\mathsf{p}_2\mathsf{Op}_{x[0]}$. As $x(0) = q_1 \circ x[0]$, $\mathsf{p}_1\mathsf{Op}_{x[0]}$ iff $\mathsf{Op}_{x(0)}$. As further, $\mathsf{p}_2 \Rightarrow \mathsf{p}_1$, $\mathsf{Op}_x \Rightarrow \mathsf{p}_1\mathsf{Op}_{x[0]}$, thus the claim.
If $o(x) = \gamma$, then $o(x) = o(x(0))$ and $x(0)[\alpha] = x[\alpha]$ by Lemma II.2.14 (i). Further, $\mathsf{Op}_{x(0)}$ iff $(\forall\alpha \lhd o(x(0)))\mathsf{p}_1\mathsf{Op}_{x(0)[\alpha]}$ iff $(\forall\alpha \lhd o(x))\mathsf{p}_1\mathsf{Op}_{x[\alpha]}$. Again, as $\mathsf{p}_2 \Rightarrow \mathsf{p}_1$, Op_x (that is $(\forall\alpha \lhd o(x))\mathsf{p}_2\mathsf{Op}_{x[\alpha]}$) implies $\mathsf{Op}_{x(0)}$.

(ii) If $o(x) \in \mathsf{Lim}(\Omega)$, then $x(0) \circ x(\alpha) = x(\alpha{+}1)$ by Lemma II.2.14 (i). Thus, $\mathsf{Op}_{x(0)} \circ \mathsf{Op}_{x(\alpha)} \Leftrightarrow \mathsf{Op}_{x(\alpha+1)}$. And if $o(x) = 1$, then $x[0] \circ x[\alpha] = x[\alpha{+}1]$ by Lemma II.2.14 (ii). Further, either $x = q_2$, and the claim reads $\mathsf{p}_1^2 \circ \mathsf{p}_1^{\alpha+1} \Rightarrow \mathsf{p}_1^{\alpha+2}$, which holds, or $x = (1, y{+}1)$ for $y \in Q_1^*$. Then $x[0] = (1, y)$, and $\mathsf{Op}_{(1,y)}\mathsf{p}_1 \Leftrightarrow \mathsf{Op}_{(1,y)}$ by Lemma II.4.6. Hence, $\mathsf{Op}_{x(0)} \circ \mathsf{Op}_{x(\alpha)} \Leftrightarrow \mathsf{p}_1\mathsf{Op}_{x[0]} \circ \mathsf{p}_1\mathsf{Op}_{x[\alpha]} \Leftrightarrow \mathsf{p}_1\mathsf{Op}_{x[0]} \circ \mathsf{Op}_{x[\alpha]} \Leftrightarrow \mathsf{p}_1\mathsf{Op}_{x[\alpha+1]} \Leftrightarrow \mathsf{Op}_{x(\alpha+1)}$.

(iii) If $o(x) = 1$, then $\deg(x[\gamma]) = 1$ and so $\breve{\mathsf{T}}_{x[\gamma]}$ iff $(\forall\alpha \lhd \gamma)\mathsf{p}_1\breve{\mathsf{T}}_{x[\alpha]}$. As further, $\mathsf{p}_1\breve{\mathsf{T}}_{x[\alpha]}$ iff $\breve{\mathsf{T}}_{x(\alpha)}$, the claim follows.

(iv) Since $\deg(x(\gamma)) = 1$, we have $\mathsf{Op}_{x(\gamma)}$ iff $(\forall\alpha \lhd o(x(\gamma)))\mathsf{p}_1\mathsf{Op}_{x(\gamma)[\alpha]}$. As $o(x) \in \mathsf{Lim}(\Omega)$, Lemma II.2.14 (i) states that $o(x(\gamma)) = \gamma$, $x(\alpha) = x(\gamma)[\alpha]$ for $\alpha \lhd \gamma$, and $x(0) \circ x(\alpha) = x(\alpha{+}1)$. Hence $\mathsf{Op}_{x(\gamma)}$ iff $(\forall\alpha \lhd \gamma)\mathsf{p}_1\mathsf{Op}_{x(\alpha)}$. Clearly, we have $(\forall\alpha{\lhd}\gamma)\mathsf{p}_1\mathsf{Op}_{x(\alpha)} \Rightarrow (\forall\alpha{\lhd}\gamma)\mathsf{Op}_{x(\alpha)}$, and since $\mathsf{Op}_{x(\alpha+1)} \Leftrightarrow \mathsf{Op}_{x(0)} \circ \mathsf{Op}_{x(\alpha)} \Rightarrow \mathsf{p}_1\mathsf{Op}_{x(\alpha)}$, also $(\forall\alpha \lhd \gamma)\mathsf{Op}_{x(\alpha)} \Rightarrow (\forall\alpha \lhd \gamma)\mathsf{p}_1\mathsf{Op}_{x(\alpha)}$ follows. □

Lemma II.4.9. $\mathsf{T}^{\epsilon} \vdash \deg(x) = 2 \wedge \breve{\mathsf{T}}_x \rightarrow \mathsf{Prog}_{\lhd}(\{\alpha : \breve{\mathsf{T}}_{x(\alpha)}\})$.

Proof Assume that $x =_{NF} (1, y) \circ z$, and so Op_x iff $\mathsf{Op}_{(1,y)} \circ \mathsf{Op}_z$ and $\mathsf{Op}_{x(\alpha)}$ iff $\mathsf{Op}_{(1,y)(\alpha)} \circ \mathsf{Op}_z$. By (i) the above lemma, $\mathsf{Op}_{(1,y)} \circ \mathsf{Op}_z \Rightarrow \mathsf{Op}_{(1,y)(0)} \circ \mathsf{Op}_z \Rightarrow \mathsf{Op}_{x(0)}$. For the successor case, note that $\deg(x(\alpha)) = 1$ and so $\breve{\mathsf{T}}_{x(\alpha)}$ is Π^1_2. Therefore, $\breve{\mathsf{T}}_x \wedge \breve{\mathsf{T}}_{x(\alpha)}$ iff $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}_z) \wedge \breve{\mathsf{T}}_{x(\alpha)}$, and Corollary II.4.7 yields $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}_{x(\alpha)})$. As $\mathsf{Op}_{(1,y)} \Rightarrow \mathsf{Op}_{(1,y)(0)}$, (ii) of the above lemma now yields $\mathsf{Op}_{(1,y)(\alpha+1)}(\breve{\mathsf{T}}_z)$, that is $\breve{\mathsf{T}}_{x(\alpha+1)}$. Now to the limit case. If $o(x) = 1$, then $x(\gamma) = x[\gamma]{+}1$. By (iii) of the above lemma we obtain that $(\forall\alpha \lhd \gamma)\breve{\mathsf{T}}_{x(\alpha)}$ iff $\breve{\mathsf{T}}_{x[\gamma]}$. Since $\deg(x[\gamma]) = 1$ and thus $\breve{\mathsf{T}}_{x[\gamma]}$ is Π^1_2, we obtain $\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}_{x[\gamma]})$ as in the successor case, which further implies $\mathsf{p}_1\breve{\mathsf{T}}_{x[\gamma]}$, that is, $\breve{\mathsf{T}}_{x(\gamma)}$. And if $o(x) \in \mathsf{Lim}(\Omega)$, then by (iv) of the above lemma, $(\forall\alpha \lhd \gamma)\breve{\mathsf{T}}_{x(\alpha)}$ iff $\breve{\mathsf{T}}_{x(\gamma)}$. □

II.4.3 The proper and provisional Definition of Op_x agree

Our next goal is to prove that the proper Definition II.4.1 agrees with the provisional Definition II.1.12. From a technical point of view, none of the results in the remainder of this section are used in the sequel.

We start with a simple observation.

Lemma II.4.10. *The following is provable in* T^ϵ.

(i) For all $x \in Q_2^*$, $\mathsf{Op}_x\mathsf{p}_1 \Rightarrow \mathsf{Op}_x$.

(ii) If $q_1 \neq x \in Q_2^*$, *then* $\mathsf{Op}_x \Rightarrow \mathsf{Op}_x\mathsf{p}_1$.

Proof Both claims are readily shown by induction using Theorem I.4.2. For (i), note that $\mathsf{p}_1 \circ \mathsf{p}_1 \Rightarrow \mathsf{p}_1$ is by Lemma I.2.12. The induction step causes no problems. For (ii), recall that if $\breve{\mathsf{T}}$ is Π^1_2if $\breve{\mathsf{T}}$ is Π^1_2, then $\mathsf{T}^\epsilon \vdash \mathsf{p}_2(\breve{\mathsf{T}}) \leftrightarrow \mathsf{p}_2\mathsf{p}_1(\breve{\mathsf{T}})$ (cf. page 39), thus also $\mathsf{Op}_{q_2} \Leftrightarrow \mathsf{p}_2\mathsf{p}_1 \Leftrightarrow \mathsf{p}_2\mathsf{p}_1^2$. To apply the I.H., just note that if $x \notin \{q_0, q_1, q_2\}$, then $q_1 \neq x[0] \in Q_2^*$. □

Recall also that Op_x^γ is an operation formed according to the convention following Corollary I.2.28, so that we have $\mathsf{Op}_x^\gamma \Leftrightarrow (\forall \beta \lhd \gamma)\mathsf{Op}_x^{\beta+1}$ (note that on the right hand side, the exponent is always a successor). That we also have $\mathsf{Op}_x^\gamma \Leftrightarrow (\forall \beta \lhd \gamma)\mathsf{Op}_x^{1+\beta}$ turns out to be a consequence of the next lemma.

Lemma II.4.11. T^ϵ *proves: for all* $x \in Q_2^*$, $\mathsf{Op}_x \circ \mathsf{Op}_x \Rightarrow \mathsf{Op}_x$.

Proof Since $\mathsf{Op}_x \Rightarrow \mathsf{p}_1$ and Op_x is an operation, we have $\mathsf{Op}_x^2 \Rightarrow \mathsf{Op}_x\mathsf{p}_1$. As further by Lemma II.4.10 (i), $\mathsf{Op}_x\mathsf{p}_1 \Rightarrow \mathsf{Op}_x$, $\mathsf{Op}_x \circ \mathsf{Op}_x \Rightarrow \mathsf{Op}_x$ follows. □

Lemma II.4.12. T^ϵ *proves: if* $0 \lhd \alpha$, *then* $(\mathsf{Op}_x)^{\alpha+1} \Rightarrow (\mathsf{Op}_x)^\alpha$.

Proof Let $A(\alpha) := 0 \lhd \alpha \wedge \mathsf{Op}_x^{\alpha+1}(\breve{\mathsf{T}}) \Rightarrow \mathsf{Op}_x^\alpha(\breve{\mathsf{T}})$. We just show the limit case. $\mathsf{Op}_x^{\gamma+1} \Leftrightarrow \mathsf{Op}_x \circ \mathsf{Op}_x^\gamma \Rightarrow (\forall \beta \lhd \gamma)\mathsf{Op}_x^{\beta+2} \Rightarrow_{IH} (\forall \beta \lhd \gamma)\mathsf{Op}_x^{\beta+1} \Leftrightarrow \mathsf{Op}_x^\gamma$. □

Corollary II.4.13. *For each* $n \in \mathbb{N}$, T^ϵ *proves: if* $0 \lhd \beta \lhd \alpha$, *then* $(\mathsf{Op}_x)^\alpha \Rightarrow (\mathsf{Op}_x)^\beta$.

Proof Let $A(\alpha) := 0 \lhd \beta \lhd \alpha \wedge \mathsf{Op}_x^{\beta+\alpha}(\breve{\mathsf{T}}) \Rightarrow \mathsf{Op}_x^\alpha(\breve{\mathsf{T}})$. We just show the successor case. If $\alpha = \beta+1$, then $A(\alpha)$ is by the previous lemma. If $\beta \lhd \alpha$ and $\mathsf{Op}_x^{\beta+\alpha+1}(\breve{\mathsf{T}})$, then the I.H. and the fact that Op_x is an operation yields $(\mathsf{Op}_x \circ \mathsf{Op}_x^\beta)(\breve{\mathsf{T}})$, and $(\mathsf{Op}_x^\beta)(\breve{\mathsf{T}})$ is by the previous Corollary. □

Corollary II.4.14. T^ϵ *proves: for all* $x \in Q_2^*$, $\mathsf{Op}_x^\gamma \Leftrightarrow (\forall \xi \lhd \gamma)\mathsf{Op}_x^{1+\xi}$.

Proof Since Op_x^γ iff $(\forall \xi \lhd \gamma)\mathsf{Op}_x^{\xi+1}$, the claim is by Lemma II.4.12. □

Lemma II.4.15. T^ϵ *proves: if* $q_2 \rightsquigarrow_r^* x$ *and* $0 \lhd \alpha$, *then*

(i) $\mathsf{p}_1(\mathsf{Op}_x)^\alpha \Rightarrow (\mathsf{p}_1\mathsf{Op}_x)^\alpha$,

(ii) $(\mathsf{p}_1\mathsf{Op}_x)^{\alpha+1} \Rightarrow \mathsf{p}_1(\mathsf{Op}_x)^\alpha$.

Proof By induction on α (in the sense of Theorem I.4.2). For both claims, we just consider the limit cases, as the successor cases follow readily from the I.H.
(i) $\mathsf{p}_1(\mathsf{Op}_x)^\gamma \Rightarrow \mathsf{p}_1(\forall\alpha \lhd \gamma)(\mathsf{Op}_x)^{\alpha+1} \Rightarrow (\forall\alpha \lhd \gamma)\mathsf{p}_1(\mathsf{Op}_x)^{\alpha+1}$, since p_1 is an operation. Using the I.H. and that $\mathsf{p}_1\mathsf{Op}_x$ is an operation, we obtain $(\mathsf{p}_1\mathsf{Op}_x)(\mathsf{p}_1(\mathsf{Op}_x)^{\alpha+1}) \Rightarrow (\mathsf{p}_1\mathsf{Op}_x)(\mathsf{p}_1\mathsf{Op}_x)^{\alpha+1}$ for each $\alpha \lhd \gamma$. By Lemma II.4.12 we obtain $(\forall\alpha\lhd\gamma)(\mathsf{p}_1\mathsf{Op}_x)^{\alpha+1}$, that is, $(\mathsf{p}_1\mathsf{Op}_x)^\gamma$.
(ii) We have that $(\mathsf{p}_1\mathsf{Op}_x)^{\gamma+1} \Rightarrow \mathsf{p}_1\mathsf{Op}_x(\forall\alpha \lhd \gamma)(\mathsf{p}_1\mathsf{Op}_x)^{\alpha+1}$. By I.H., and as by Lemma II.4.2, $\mathsf{p}_1\mathsf{Op}_x\mathsf{p}_1 \Rightarrow \mathsf{p}_1$, we obtain for each $0\lhd\alpha\lhd\gamma$ that $(\mathsf{p}_1\mathsf{Op}_x)(\mathsf{p}_1\mathsf{Op}_x)^{\alpha+1} \Rightarrow (\mathsf{p}_1\mathsf{Op}_x)\mathsf{p}_1(\mathsf{Op}_x)^\alpha \Rightarrow \mathsf{p}_1(\mathsf{Op}_x)^\alpha$. Further, if $A(u)$, $B(u)$ are open L_2-sentences so that $\forall\alpha(A(\alpha) \to B(\alpha))$, then also $\forall\alpha A(\alpha) \to \forall\alpha B(\alpha)$, and thus for each operation Op, $\mathsf{Op}(\forall\alpha A(\alpha)) \to \mathsf{Op}(\forall\alpha B(\alpha))$. Therefore, we obtain $\mathsf{p}_1\mathsf{Op}_x(\forall\alpha \lhd \gamma)\mathsf{p}_1(\mathsf{Op}_x)^{1+\alpha}$, and we further conclude $\mathsf{p}_1(\forall\alpha \lhd \gamma)\mathsf{Op}_x\mathsf{p}_1(\mathsf{Op}_x)^{1+\alpha} \Rightarrow \mathsf{p}_1(\forall\alpha \lhd \gamma)\ \mathsf{Op}_x(\mathsf{Op}_x)^{1+\alpha} \Rightarrow \mathsf{p}_1(\forall\alpha \lhd \gamma)(\mathsf{Op}_x)^{\alpha+1} \Rightarrow \mathsf{p}_1(\mathsf{Op}_x)^\gamma$. □

Next, two auxiliary properties of the operations Op_x^+.

Lemma II.4.16. T^ϵ *proves: if* $x \in Q_1^*$, *then* $\mathsf{Op}_x^+\mathsf{p}_1 \Leftrightarrow \mathsf{Op}_x^+\mathsf{p}_1^2$.

Proof With $\mathsf{p}_1^2 \Rightarrow \mathsf{p}_1$ also $\mathsf{Op}_x^+\mathsf{p}_1^2 \Rightarrow \mathsf{Op}_x^+\mathsf{p}_1$. As further, $\mathsf{Op}_x^+ \Rightarrow \mathsf{p}_1$, $\mathsf{Op}_x^+\mathsf{p}_1(\breve{\mathsf{T}})$ yields $\mathsf{p}_1^2(\breve{\mathsf{T}})$ which is Π_2^1, hence $\mathsf{Op}_x^+\mathsf{p}_1^2(\breve{\mathsf{T}})$ is by Lemma II.4.5. □

Lemma II.4.17. T^ϵ *proves: for all* $x \in Q_1^*$, $\mathsf{Op}_x^+\mathsf{p}_1 \Leftrightarrow \mathsf{Op}_{(1,x)}$.

Proof If $x = (1, q_0)$, then $\mathsf{Op}_x^+\mathsf{p}_1$ is $\mathsf{p}_2\mathsf{p}_1$ is $\mathsf{Op}_{(1,x)}$. Otherwise, $(1, x)[\alpha] = (1, x[\alpha])$ and $\deg(x) = 2$. Then, $\mathsf{Op}_x^+\mathsf{p}_1 \Leftrightarrow (\forall\xi \lhd \gamma)\mathsf{p}_2\mathsf{Op}_{x[\xi]}^+\mathsf{p}_1 \Leftrightarrow_{IH} (\forall\xi \lhd \gamma)\mathsf{p}_2\mathsf{Op}_{(1,x[\xi])} \Leftrightarrow (\forall\xi \lhd \gamma)\mathsf{p}_2\mathsf{Op}_{(1,x)[\xi]} \Leftrightarrow \mathsf{Op}_{(1,x)}$. □

Now we can prove the aforementioned equivalence of our definitions of Op_x. Most of the work is done by proving the next two lemmas.

Lemma II.4.18. T^ϵ *proves the following: if* $0 \lhd \beta$, *then*

$$\mathsf{p}_2^\beta\mathsf{p}_1 \circ (\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma \Leftrightarrow (\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma.$$

Proof Recall that $(\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma \Leftrightarrow (\forall\xi \lhd \gamma)(\mathsf{p}_2^\beta\mathsf{p}_1)^{1+\xi}$ (cf. Corollary II.4.14). To show the $\Rightarrow$-direction, fix an $\eta \rhd 0$. Since $\mathsf{p}_2^\beta\mathsf{p}_1$ iff $\mathsf{p}_2^\beta\mathsf{p}_1^2$ (cf. Lemma II.4.10), and $(\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma \Rightarrow (\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^{\eta+1} \Rightarrow_{L.II.4.15} \mathsf{p}_1(\mathsf{p}_2^\beta\mathsf{p}_1)^\eta$, we have $\mathsf{p}_2^\beta\mathsf{p}_1 \circ (\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma \Rightarrow (\mathsf{p}_2^\beta\mathsf{p}_1)^{\eta+1} \Rightarrow_{L.II.4.12} (\mathsf{p}_2^\beta\mathsf{p}_1)^\eta$. For the converse direction observe that $(\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma \Rightarrow (\forall\xi \lhd \gamma)(\mathsf{p}_2^\beta\mathsf{p}_1)^{\xi+1} \Rightarrow (\forall\xi \lhd \gamma)\mathsf{p}_1(\mathsf{p}_2^\beta\mathsf{p}_1)^\xi \Rightarrow (\forall\xi \lhd \gamma)(\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^\xi$. Let $C := (\forall\xi \lhd \gamma)(\mathsf{p}_1\mathsf{p}_2^\beta\mathsf{p}_1)^\xi(\breve{\mathsf{T}})$, which is Π_2^1, and note that $(\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma(\breve{\mathsf{T}})$ implies $\mathsf{p}_2^\beta(\mathsf{p}_1(\breve{\mathsf{T}})) \wedge C$. Hence Lemma II.4.5 further yields $\mathsf{p}_2^\beta(C)$, therefore also $\mathsf{p}_2^\beta\mathsf{p}_1(C)$, that is, $\mathsf{p}_2^\beta\mathsf{p}_1 \circ (\mathsf{p}_2^\beta\mathsf{p}_1)^\gamma(\breve{\mathsf{T}})$. □

Lemma II.4.19. T^ϵ *proves the following: if* $x \in Q_1^*$, *then* *(i)* $\mathsf{Op}_{(m+1,x)} \Leftrightarrow (\mathsf{Op}_{(1,x)})^{m+1}$, *and* $\mathsf{Op}_{(\gamma+m+1,x)} \Leftrightarrow (\mathsf{Op}_{(1,x)})^{\gamma+m}$.

(ii) $\mathsf{Op}_{(\gamma,x)} \Leftrightarrow (\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\gamma}$.

Proof Using Theorem I.4.2. We let $A(\alpha)$ so that $A(m{+}1)$ and $A(\gamma{+}m{+}1)$ express (i), and $A(\gamma)$ expresses (ii). First, we show $A(\gamma{+}1)$, i.e., $\mathsf{Op}_{(\gamma,x)} \Leftrightarrow (\mathsf{Op}_{(1,x)})^{\gamma+1}$. By I.H. we have $\mathsf{Op}_{(\gamma,x)}(\breve{\mathsf{T}})\restriction X \Leftrightarrow (\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\gamma}(\breve{\mathsf{T}})\restriction X$. As $\mathsf{Op}_{(1,x)}$ is an operation, $\mathsf{Op}_{(1,x)} \circ \mathsf{Op}_{(\gamma,x)} \Leftrightarrow \mathsf{Op}_{(1,x)} \circ (\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\gamma}$ follows. By Lemma II.4.4 (iv) and Lemma II.4.18, we obtain $\mathsf{Op}_{(\gamma+1,x)} \Leftrightarrow (\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\gamma}$. $A(1)$ holds trivially, and $A(m{+}1)$ and $A(\gamma{+}m{+}2)$ are directly from the I.H.

Now we show $A(\gamma)$. Using the definition of $\mathsf{Op}_{(\gamma,x)}$ and Corollary II.4.14, this amounts to show that

$$(\forall\xi \lhd \gamma)\mathsf{p}_1\mathsf{Op}_{(1+\xi,x)} \Leftrightarrow (\forall\xi \lhd \gamma)(\mathsf{p}_1\mathsf{Op}_{(1,x)})^{1+\xi}.$$

To show that $\Rightarrow$-direction, fix a $\eta \lhd \gamma$ with $\eta \rhd 0$. $(\forall\xi \lhd \gamma)\mathsf{p}_1\mathsf{Op}_{(1+\xi,x)}$ entails $\mathsf{p}_1\mathsf{Op}_{(\eta+1,x)}$. Using the I.H. yields $\mathsf{p}_1(\mathsf{Op}_{(1,x)})^{\eta}$, and $(\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\eta}$ follows by Lemma II.4.15. For the converse direction, also fix an $\eta \lhd \gamma$ with $\eta \rhd 0$. Note that $(\forall\xi\lhd\gamma)(\mathsf{p}_1\mathsf{Op}_{(1,x)})^{1+\xi}$ entails $(\mathsf{p}_1\mathsf{Op}_{(1,x)})^{\eta+1}$. Lemma II.4.15 yields $\mathsf{p}_1(\mathsf{Op}_{(1,x)})^{\eta}$. Using the I.H. and possibly Lemma II.4.4 (iv) and Lemma II.4.12 yields $\mathsf{p}_1\mathsf{Op}_{(\eta,x)}$. □

Theorem II.4.20.

(i) $\mathsf{Op}_{(\alpha,q_0)} \Leftrightarrow \mathsf{p}_1^{\alpha}$ *and* $\mathsf{Op}^{+}_{(\alpha,q_0)} \Leftrightarrow \mathsf{p}_2^{\alpha}$.

(ii) $\mathsf{Op}_{(\alpha,(\gamma,q_0)^-)} \Leftrightarrow (\mathsf{Op}_{(1,(\gamma,q_0)^-)})^{\alpha}$.

(iii) $\mathsf{Op}_{(n,(\beta,q_0))} \Leftrightarrow (\mathsf{Op}^{+}_{(\beta,q_0)}\mathsf{p}_1)^{n}$ *and* $\mathsf{Op}_{(\gamma+n+1,(\beta,q_0))} \Leftrightarrow (\mathsf{Op}^{+}_{(\beta,q_0)}\mathsf{p}_1)^{\gamma+n}$.

(iv) $\mathsf{Op}_{(\gamma,(\beta,q_0))} \Leftrightarrow (\mathsf{p}_1\mathsf{Op}^{+}_{(\beta,q_0)}\mathsf{p}_1)^{\gamma}$.

Proof (i) By definition and our convention of how to read p_1^{α} and p_2^{α}.

(ii) Let $v := (\gamma, q_0)^-$ and $A(\alpha) := \mathsf{Op}_{(\alpha,v)}(\breve{\mathsf{T}}) \leftrightarrow (\mathsf{Op}_{(1,v)})^{\alpha}(\breve{\mathsf{T}})$. If $\alpha = \alpha'{+}1$, then $\mathsf{Op}_{(\alpha,v)}$ iff $\mathsf{Op}_{(1,v)} \circ \mathsf{Op}_{(\alpha',v)}$, and $A(\alpha)$ follows using the I.H. If $\alpha =: \gamma$ is a limit, then $\mathsf{Op}_{(\gamma,v)}$ iff $(\forall\xi\lhd\gamma)\mathsf{p}_1\mathsf{Op}_{(1+\xi,v)}$. The I.H. implies that $\mathsf{Op}_{(\gamma,v)}$ iff $(\forall\xi\lhd\gamma)\mathsf{p}_1(\mathsf{Op}_{(1,v)})^{1+\xi}$. As $(\mathsf{Op}_{(1,v)})^{1+\xi}$ is Π^1_2, for $0 \lhd \xi \lhd \gamma$, $\mathsf{p}_1(\mathsf{Op}_{(1,v)})^{\xi} \Rightarrow (\mathsf{Op}_{(1,v)})^{\xi}$, and $\mathsf{Op}_{(1,v)})^{\xi+1} \Rightarrow \mathsf{p}_1(\mathsf{Op}_{(1,v)})^{\xi}$, hence $A(\gamma)$ follows.

(iii) and (iv) are by Lemma II.4.19. □

II.5 Modular ordinal analysis at work

In this section, we prove one of the main results of this theses for the case $N_0 = 2$. We fix $\breve{\mathsf{T}} := (\mathsf{ACA})$ and $g(\alpha) := \omega^{1+\alpha}$, and show in particular that the following is provable in T^ϵ: for each $x \in Q_2$, $\breve{\mathsf{T}}_x$ implies $\mathsf{Wo}_\triangleleft(g_{1+x^H}(\alpha))$, under the assumption that $\mathsf{Wo}_\triangleleft(\alpha)$ and $\mathsf{TI}_\triangleleft(\mathcal{C}_x, \alpha)$ for a suitable class $\mathcal{C}_x$. We refer to this statement as "T_x proves $g_{1+x^H} := H_{1+x^H}(g)$".

Remark II.5.1. *The theory ACA_0 can be presented as $\tilde{\mathsf{p}}_1(\Pi^0_1\text{-}\mathsf{CA}_0^-)$, where $\tilde{\mathsf{p}}_1$ is a variant of the operation p_1, and $\Pi^0_1\text{-}\mathsf{CA}_0^-$ is a finitely axiomatized version of the theory that extends T^ϵ by Π^0_1-comprehension with a positive set parameters U and set induction (see Appendix, Section 2). It can be argued that "$\Pi^0_1\text{-}\mathsf{CA}_0^-$ proves $g(\alpha)$", which is the underlying reason why "$\tilde{\mathsf{p}}_1(\Pi^0_1\text{-}\mathsf{CA}_0^-)$ proves $\mathsf{it}(g)$", or put differently, that "$\breve{\mathsf{T}}'_{q_1}$ proves g_{q_1}" for $\breve{\mathsf{T}}' := \Pi^0_1\text{-}\mathsf{CA}_0^-$.*

With a slightly more general notion of operation at hand, we would prove that for all $x \in Q_2^*$, "$\mathsf{Op}_x(\Pi^0_1\text{-}\mathsf{CA}_0^-)$ proves g_{x^H}", thus we had also that "$\mathsf{Op}_{1+x}(\Pi^0_1\text{-}\mathsf{CA}_0^-)$ proves $g_{(1+x)^H}$" (where $1{+}x := x{+}1$ if $x <_2 (\omega, q_0)$, and $1{+}x := x$ otherwise). The point is that $\mathsf{Op}_{1+x}(\Pi^0_1\text{-}\mathsf{CA}_0^-)$ iff $\mathsf{Op}_x(\mathsf{ACA}_0)$. As $x^H = x$ for $x <_2 (\omega, q_0)$, and thus $(1{+}x)^H = 1{+}x^H$, this explains why we show that "T_x proves g_{1+x^H}".

II.5.1 The definition of "Op_x proves H_{x^H}"

The above discussion suggest to think of $1{+}x^H$ as x^h, where x^h is the name of the function which corresponds to the theory T_x.

Definition II.5.2. *For $x \in Q_2$,* $x^h := \begin{cases} (x{+}1)^H & : x <_2 (\omega, q_0), \\ x^H & : \textit{else.} \end{cases}$

Thus, $x^h \neq x^H$ only if x is of the form (n, q_0). In particular, $x^h = x^H$ if $o(x) \in \mathsf{Lim}(\Omega)$ or $\deg(x) > 1$. As $(x \circ y)^H = x^H \circ y^H$ by Lemma II.2.18, also $(x \circ y)^h = x^H \circ y^h$. We show that for each $x \in Q_2^*$, "Op_x proves H_{x^H}", which states essentially, that if $x \circ z \in Q_2$ and "$\breve{\mathsf{T}}_z$ proves g_{z^h}", then "$\mathsf{Op}_x(\breve{\mathsf{T}}_z)$ proves $H_{x^H}(g_{z^h})$". Observe that $\mathsf{Op}_x(\breve{\mathsf{T}}_z)$ iff $\breve{\mathsf{T}}_{x \circ z}$, and that $H_{x^H}(g_{z^h}) = g_{(x \circ z)^h}$.

We begin by defining when a theory proves a function, and when an operation proves a functional. Thereto, we fix the auxiliary classes $(\mathcal{C}_x : x \in Q_2)$. Since $\deg(x) = 2$ implies $\deg(x(\alpha)) = 1$ and thus $\breve{\mathsf{T}}_{x(\alpha)}$ is Π^1_2 (cf. Definition II.2.11 and Lemma II.4.3), we have the following: if $\deg(x) = 1$, then $\mathcal{C}_x$ is Π^1_1, and if $\deg(x) = 2$, then $\mathcal{C}_x$ is Π^1_2.

Definition II.5.3. $\mathcal{C}_x := \{\alpha : [(\deg(x) \leq 1 \wedge \mathsf{Wo}_\triangleleft(g_{x^h}(\alpha))] \vee [\deg(x) = 2 \wedge \breve{\mathsf{T}}_{x(\alpha)}]\}$.

Next, we say when "T_x proves g_y".

Definition II.5.4. *We say that* T_x *proves* g_y*, if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(x,y)$*, where*

$$\mathsf{Prv}_0(x,y) := \breve{\mathsf{T}}_x \wedge y \in Q_2^H \to \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_x,\alpha) \to \mathsf{Wo}_\lhd(g_y(\alpha))].$$

Since we just show that T_x proves g_{x^h}, we only need the one-parameter version $\mathsf{Prv}_0(x) := \mathsf{Prv}_0(x,x^h)$. Further, we define when an operation proves a functional. The formal definitions are given below, the idea (neglecting some details) is the following. If $\mathsf{Prv}_1(x)$ and $\mathsf{Prv}_0(y)$, then also $\mathsf{Prv}_0(x\circ y)$. Since things are set up so that $\mathsf{Prv}_0(q_0)$, i.e. $\breve{\mathsf{T}}$ proves $g_{q_0^h}$ (ACA_0 proves $\mathsf{it}(g)$), $\mathsf{Prv}_1(x)$ implies $\mathsf{Prv}_0(x)$. More verbosely, $\mathsf{Prv}_1(x)$ states that if "$\breve{\mathsf{T}}_y$ proves g_{y^h}", then "$\mathsf{Op}_x(\breve{\mathsf{T}}_y)$ proves $H_{x^H}(g_{y^h})$". Similarly, $\mathsf{Prv}_2(x)$ states that if "$\mathsf{Op}_y(\breve{\mathsf{T}}_z)$ proves $H_{y^H}(g_{z^h})$", then "$\mathsf{Op}^+_x(\mathsf{Op}_y)(\breve{\mathsf{T}}_z)$ proves $H^+_{x^*}[H_{y^H},g_{z^h}]$", where $x^* := x^H + \mathsf{corr}(x)$.

To avoid a logic of partial terms, we deal with the partial function $\circ$ as follows. We assume that "undefined" is some fixed natural number n that does not coded a name. For this n, let $g_n := g$. Further, as by its definition, $\breve{\mathsf{T}}_x \to x \in Q_2$, $\breve{\mathsf{T}}_n \leftrightarrow \bot$.

Definition II.5.5. *We fix the following formulas.*

$$\begin{aligned}
\mathsf{Prv}_0(x) &:= \breve{\mathsf{T}}_x \to \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_x,\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))],\\
\mathsf{Prv}_1(x) &:= \forall y[\mathsf{prv}_0(y) \to \mathsf{Prv}_0(x\circ y)],\\
\mathsf{Prv}_2(x) &:= \forall y[\mathsf{prv}_1(1,y) \to \mathsf{Prv}_1(1,x\circ y)].
\end{aligned}$$

Further, for $n \in \{0,1,2\}$*,* $\mathsf{prv}_n(x) := \forall X \mathsf{Prv}_n(x)\restriction X$*.*

Moreover, we say that Op_x *proves* H_{x^H}*, if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_1(x)$*, and that* Op^+_x *proves* $H^+_{x^*}$*, if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_2(x)$ *(where* $y^* := y^H + \mathsf{corr}(y)$*).*

Let us discuss this definition. Firstly, we point out that by definition of $\breve{\mathsf{T}}_u$ (cf. Definition II.4.1 and Lemma II.4.2), $\breve{\mathsf{T}}_u \to u \in Q_2$, and therefore, $\mathsf{Prv}_0(u)$ is trivially true if $u \notin Q_2$. Consequently, $\mathsf{Prv}_1(x)$ iff $\forall y[x\circ y \in Q_2 \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(x\circ y)]$, and $\mathsf{Prv}_2(x)$ iff $\forall y[x\circ y \in Q_1 \wedge \mathsf{prv}_1(1,y) \to \mathsf{Prv}_1(1,x\circ y)]$. Often, we use these equivalent forms to focus on the non-trivial instances of these definitions. Moreover, note that $\mathsf{Prv}_1(q_0)$, i.e. $\forall y[\mathsf{prv}_0(y) \to \mathsf{Prv}_0(y)]$ is not provable as $\mathsf{prv}_0(y)$ may hold trivially if there are no ω-models of $\breve{\mathsf{T}}_y$.

Secondly, we elaborate on Op_x proves H_{x^H}. For this purpose, let

$$\mathsf{j}_{\mathcal{C}_x}(g_{x^h}) := \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_x,\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))]$$

express that with $\mathsf{TI}_\lhd(\mathcal{C}_x,\alpha)$ at hand, we can jump from $\mathsf{Wo}_\lhd(\alpha)$ to $\mathsf{Wo}_\lhd(g_{x^h}(\alpha))$. Now assume $\mathsf{T}_z \vdash \mathsf{j}_{\mathcal{C}_z}(g_{z^h})$. Then, $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}_z \to \mathsf{j}_{\mathcal{C}_z}(g_{z^h})$, that is, $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(z)$, and thus also $\mathsf{T}^\epsilon \vdash \mathsf{prv}_0(z)$ by Lemma I.1.9. Next, we further assume $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_1(y)$ (so again, also $\mathsf{T}^\epsilon \vdash \mathsf{prv}_1(y)$). The definition of $\mathsf{Prv}_1(y)$ implies $\mathsf{Op}_y(\breve{\mathsf{T}}_z) \to \mathsf{j}_{\mathcal{C}_{y \circ z}}(H_{y^H}(g_{z^h}))$ (as $\mathsf{Op}_x(\breve{\mathsf{T}}_y)$ iff $\breve{\mathsf{T}}_{x \circ y}$ and $H_{x^H}(g_{y^h}) = g_{x^H \circ y^h} = g_{(x \circ y)^h}$). This illustrates why we read $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_1(x)$ as "Op_y proves H_{y^H}".

Thirdly, we have a closer look at $\mathsf{Prv}_2(x)$. The point of letting $y^* := y^H + \mathsf{corr}(y)$ is that now $(1, y)^H = (1, y^*)$ (cf. Definition II.2.17). Further, it is easily verified that for $x, y \in Q_1$, $(x \circ y)^* = x^* \circ y^*$. Next assume $\mathsf{Prv}_2(x)$ and that $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_1(1, y)$ (and still $\mathsf{T}_z \vdash \mathsf{j}_{\mathcal{C}_z}(g_{z^h})$). The definition of $\mathsf{Prv}_2(x)$ implies that $\mathsf{Prv}_1(1, x \circ y)$. As we have seen above, this yields $\mathsf{Op}_{(1,x\circ y)}(\breve{\mathsf{T}}_z) \to \mathsf{j}_{\mathcal{C}_{(1,x\circ y)\circ z}}(H_{(1,x\circ y)^H}(g_{z^h}))$. Using basic properties of operations and functionals (cf. Lemma II.4.4 (iii) and (v), and Definition II.1.5) we see that $\mathsf{Op}_{(1,x\circ y)}$ iff $\mathsf{Op}^+_x \circ \mathsf{Op}^+_y \mathsf{p}_1$ iff $\mathsf{Op}^+_x \circ \mathsf{Op}_{(1,y)}$, and $H_{(1,x\circ y)^H} = (H^+_{x^*} \circ H^+_{y^*})(\mathsf{it}) = H^+_{x^*}(H_{(1,y)^H})$. Summing up, we have that

$$\mathsf{Op}^+_x(\mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}_z)) \to \mathsf{j}_{\mathcal{C}_{(1,x\circ y)\circ z}}(H^+_{x^*}(H_{(1,y)^H})(g_{z^h})).$$

Therefore, we read $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_2(x)$ as Op^+_x proves $H^+_{x^*}$. For instance, we will see that $\mathsf{Op}^+_{(n,q_0)}$ proves $H^+_{(n,q_0)}$, and that $\mathsf{Op}^+_{(\omega,q_0)}$ proves $H^+_{(\omega+1,q_0)}$. In other words, we have that p_2^{n+1} proves It^{n+1}, and p_2^ω proves $\mathsf{It}^{\omega+1}$.

II.5.2 Elementary properties of $\mathsf{Prv}_m(x)$ ($m \in \{0, 1, 2\}$)

Since by the very form of the formula $\mathsf{Prv}_0(x)$, $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(x) \vee (\mathsf{ACA})$, it suffices to show that $\mathsf{ACA}_0 \vdash \mathsf{Prv}_0(x)$: then also $\mathsf{T}^\epsilon \vdash (\mathsf{ACA}) \to \mathsf{Prv}_0(x)$, and since $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(x) \vee (\mathsf{ACA})$, $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(x)$ follows. The same holds true for $\mathsf{Prv}_1(x)$ and $\mathsf{Prv}_2(x)$. In fact, we even have that for $m \in \{0, 1, 2\}$, $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_m(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_{\leadsto^*}(x))$. This immediately follows from the next lemma which unwinds the definition of $\mathsf{Prv}_2(x)$.

Lemma II.5.6. $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_2(x) \leftrightarrow \forall y, z[\mathsf{prv}_1(1, y) \wedge \mathsf{prv}_0(z) \to \mathsf{Prv}_0((1, (x \circ y)) \circ z)]$.

As $\breve{\mathsf{T}}_x$ implies $\mathsf{Wo}_{\leadsto^*}(x) \wedge (\mathsf{ACA})$, and by inspection of Definition II.5.5 and the above lemma, for each $m \in \{0, 1, 2\}$, $\mathsf{Prv}_m(x)$ is equivalent to a formula of the form $\breve{\mathsf{T}}_x \to B$, we have the Lemma below, which puts us into the position to prove e.g. $(\forall x \in Q_2^*)\mathsf{Prv}_1(x)$ by transfinite induction along $\leadsto^*$ using Theorem I.4.2.

Lemma II.5.7. *For each $n \in \{0, 1, 2\}$, $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_n(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wo}_{\leadsto^*}(x))$.*

One more thing we like the recall is that if e.g. $\mathsf{ACA}_0 \vdash A \wedge b \to \mathsf{Prv}_m(x)$ (where $b = \forall X B{\restriction}X$), then as $\mathsf{T}^\epsilon \vdash (\mathsf{ACA}) \vee \mathsf{Prv}_m(x)$, we also have $\mathsf{T}^\epsilon \vdash A \to \mathsf{Prv}_m(x)$, and thus $\mathsf{T}^\epsilon \vdash a \wedge \forall X b{\restriction}X \to \mathsf{prv}_m(x)$. By Lemma I.4.5, $\mathsf{ACA}_0 \vdash a \wedge b \to \mathsf{prv}_m(x)$ follows. We refer to this as the "small variant" of $\mathsf{ACA}_0 \vdash A \wedge b \to \mathsf{Prv}_m(x)$. For instance, we have the following.

Lemma II.5.8. *The following is provable in ACA_0:*

(i) $x \circ y \in Q_2^* \wedge \mathsf{prv}_1(x) \wedge \mathsf{prv}_0(y) \to \mathsf{prv}_0(x \circ y)$.

(ii) $x \circ y \in Q_1^* \wedge \mathsf{prv}_2(x) \wedge \mathsf{prv}_1(1, y) \to \mathsf{prv}_1(1, x \circ y)$.

Proof Note that the assumptions $x \circ y \in Q_2^*$ and $x \circ y \in Q_1^*$ are superfluous, as e.g. $\mathsf{prv}_0(x \circ y)$ entails trivially $x \circ y \in Q_2^*$. However, we like to focus on the relevant instances. (i) and (ii) are the "small variants" of $\mathsf{Prv}_1(x) \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(x \circ y)$ and $\mathsf{Prv}_2(x) \wedge \mathsf{prv}_1(1, y) \to \mathsf{Prv}_1(1, x \circ y)$, which hold by definition of $\mathsf{Prv}_1(x)$ and $\mathsf{Prv}_2(x)$, respectively. $\square$

Also the following simple observations are used tacitly in the sequel.

Lemma II.5.9. *The following is provable in* ACA_0*:*

(i) $x \circ y \in Q_2^* \wedge \mathsf{Prv}_1(x) \wedge \mathsf{prv}_1(y) \to \mathsf{Prv}_1(x \circ y)$,

(ii) $x \circ y \in Q_1^* \wedge \mathsf{Prv}_2(x) \wedge \mathsf{prv}_2(y) \to \mathsf{Prv}_2(x \circ y)$.

Proof (i) $\mathsf{Prv}_1(x \circ y)$ holds, if $\mathsf{prv}_0(z)$ implies $\mathsf{Prv}_0(x \circ y \circ z)$. To verify the claim, assume $\mathsf{prv}_0(z)$, and further $\mathsf{Prv}_1(x)$ and $\mathsf{prv}_1(y)$. By (i) of the above lemma, $\mathsf{prv}_1(y)$ and $\mathsf{prv}_0(z)$ yield $\mathsf{prv}_0(y \circ z)$, and $\mathsf{Prv}_0(x \circ y \circ z)$ follows from $\mathsf{Prv}_1(x)$. (ii) is shown analogously. $\square$

Finally, we move a last technicality out of the way, concerning the interplay of $\cdot[\alpha]$, $\cdot(\alpha)$ and $\cdot^H$. We will use (i) in the proof of Lemma II.5.13 (ii), and (ii) in the proof of Lemma II.5.15.

Lemma II.5.10. *Let* $z \in Q_2^*$*. Then we have the following.*

(i) *If* $\deg(z) = 1$ *and* $o(z) = \gamma$*, then* $g_{z^H[\alpha]} \trianglelefteq g_{(z[\alpha])^H}$.

(ii) *if* $\deg(z) = 2$*, then* $g_{z^H[\alpha]} \trianglelefteq g_{(z(\alpha))^H}$.

Proof By Lemma II.2.19, we have in case (i) $z^H[\alpha] \rightsquigarrow^* (z[\alpha])^H$, and in case (ii) $z^H[\alpha] \rightsquigarrow^* z(\alpha))^H$. Thus the claim follows by Lemma II.3.2. $\square$

II.5.3 A sketch of the proof

In this subsection, we sketch how we prove $(\forall x \in Q_2^*)\mathsf{Prv}_1(x)$. Thereby, we neglect the difference between $\mathsf{Prv}_m(x)$ and $\mathsf{prv}_m(x)$ ($m \in \{0, 1, 2\}$), and we occasionally embezzle some details. In the next subsection, this sketch is then turned into a correct proof.

For this sketch, we are pretending that for $m \in \{0, 1\}$, $\mathsf{Prv}_m(x) \wedge \mathsf{Prv}_m(y)$ implies $\mathsf{Prv}_m(x \circ y)$ (our current reading of Lemma II.5.9). Further, we assume for the moment the following.

(a) if $\deg(x) = 1$ then $(\forall\alpha \lhd o(x))\mathsf{Prv}_1(x[\alpha])$ implies $\mathsf{Prv}_1(x)$,

(b) if $\deg(x) = 2$, then $\forall\alpha\mathsf{Prv}_1(x(\alpha))$ implies $\mathsf{Prv}_1(x)$,

(c) if $\deg(x) = 1$, then $\breve{\mathsf{T}}_x \wedge (\forall\alpha \lhd o(x))\mathsf{Prv}_0(x[\alpha]) \to \mathsf{Prog}_\lhd(\mathcal{C}_x)$.

(a) and (b) provide means to conclude $\mathsf{Prv}_1(x)$, depending on the degree of the name. We will justify these claims below. Moreover, we rely on (c) which hides technical details (cf. Lemma II.5.13).

In order to obtain $(\forall x \in Q_2^*)\mathsf{Prv}_1(x)$, we first show $\mathsf{Prv}_1(1, q_0)$ and $\mathsf{Prv}_2(1, q_0)$. Then, we prove by induction on β that $\mathsf{Prv}_2(1{+}\beta, q_0)$, that is, $(\forall y \in Q_1^*)\mathsf{Prv}_2(y)$. Finally, an easy induction along $\rightsquigarrow^*$ yields $(\forall x \in Q_2^*)\mathsf{Prv}_1(x)$; the possible cases are discussed below.

(i) $\deg(x) = 1 \wedge x \neq (1, q_0)$. By I.H., $(\forall\alpha \lhd o(x))\mathsf{Prv}_1(x[\alpha])$, and $\mathsf{Prv}_1(x)$ is by (a).

(ii) $\deg(x) = 2$. Then $x =_{NF} (1, (1{+}\beta, q_0)) \circ y$. As $y \rightsquigarrow^* x$ by Lemma II.2.16 (iv), the I.H. yields $\mathsf{Prv}_1(y)$. Further, $\mathsf{Prv}_2(1{+}\beta, q_0)$ and $\mathsf{Prv}_1(1, q_0)$ yield $\mathsf{Prv}_1(1, (1{+}\beta, q_0))$ by definition of $\mathsf{Prv}_2(\cdot)$, which together with $\mathsf{Prv}_1(y)$ yields $\mathsf{Prv}_1(x)$.

Now we explain how to obtain $\mathsf{Prv}_1(1, q_0)$. Note that $\mathsf{Prv}_1(1, q_0)$ states that $\mathsf{Prv}_0(x)$ implies $\mathsf{Prv}_0(x{+}1)$, or in other words, if T_x proves g_{x^h}, then $\breve{\mathsf{T}}_{x+1}$ proves g_{x^h+1} (i.e. $\mathsf{p}_1(\breve{\mathsf{T}}_x)$ proves $\mathsf{it}(g_{x^h})$). To show $\mathsf{Prv}_1(1, q_0)$, we hence assume $\mathsf{Prv}_0(x)$ and aim for $\mathsf{Prv}_0(x{+}1)$. For that, we further assume $\breve{\mathsf{T}}_{x+1}$, $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{TI}_\lhd(\mathcal{C}_{x+1}, \alpha)$, and verify that $\mathsf{Wo}_\lhd(g_{x^h+1}(\alpha))$. By (c), $\breve{\mathsf{T}}_{x+1}$ and $\mathsf{Prv}_0(x)$ yield $\mathsf{Prog}_\lhd(\mathcal{C}_{x+1})$. Together with $\mathsf{TI}_\lhd(\mathcal{C}_{x+1}, \alpha)$, we conclude $\alpha \in \mathcal{C}_{x+1}$, which says $\mathsf{Wo}_\lhd(g_{x^h+1}(\alpha))$ due to the definition of $\mathcal{C}_{x+1}$.

More work goes into $\mathsf{Prv}_2(1, q_0)$: we have to verify that $\mathsf{Prv}_1(1, y) \to \mathsf{Prv}_1(1, y{+}1)$. To do so, we prove by induction on α that $\mathsf{Prv}_1(1, y)$ implies $\forall\alpha\mathsf{Prv}_1(1{+}\alpha, y)$. Then, as $(1{+}\alpha, y){+}1 = (1, y{+}1)(\alpha)$, and since we already have $\mathsf{Prv}_1(q_0)$, $\forall\alpha\mathsf{Prv}_1((1, y{+}1)(\alpha))$ is readily obtained, and $\mathsf{Prv}_1(1, y{+}1)$ is by (b). Back to the induction on α: as $\mathsf{Prv}_1(1, y) \wedge \mathsf{Prv}_1(\alpha, y)$ implies $\mathsf{Prv}_1(\alpha{+}1, y)$, the successor case is immediate, and the limit case is by (a), since $\deg(\gamma, y) = 1$, and $(\gamma, y)[\alpha] = (1{+}\alpha, y)$.

The induction on β which establishes $\forall\beta\mathsf{Prv}_2(1{+}\beta, q_0)$ makes use of a characteristic two-step approximation in the limit case (again, the successor step is for free). It is assumed that $(\forall\beta \lhd \gamma)\mathsf{Prv}_2(1{+}\beta, q_0)$, and we aim for $\mathsf{Prv}_2(\gamma, q_0)$. For that, we further assume that $\mathsf{Prv}_1(1, y)$, and verify $\mathsf{Prv}_1(1, z)$ for $z := (\gamma, q_0) \circ y$. We point out that $\deg(1, z) = 2$. The first step uses (a). To be in position to do so, we consider the name $(1, z^-)$ which is of degree one. Note that $(1, z)[\alpha] = (1, z^-)[\alpha]$. The assumptions $(\forall\beta \lhd \gamma)\mathsf{Prv}_2(1{+}\beta, q_0)$ and $\mathsf{Prv}_1(1, y)$ yield $(\forall\beta \lhd \gamma)\mathsf{Prv}_1((1, z^-)[\beta])$. Now (a) yields $\mathsf{Prv}_1(1, z^-)$. For the second step, observe that another induction on

α yields $\forall\alpha\mathsf{Prv}_1(1{+}\alpha, z^-)$: the successor case causes no problems, and the limit case is again by (a), as $\deg(\gamma', z^-) = 1$ and $(\gamma', z^-)[\alpha] = (1{+}\alpha, z^-)$. Hence we have $\forall\alpha\mathsf{Prv}_1((1,z)(\alpha))$, as $(1,z)(\alpha) = (1{+}\alpha, z^-)$, and (b) implies $\mathsf{Prv}_1(1,z)$. Hence $\mathsf{Prv}_2(\gamma, q_0)$.

Next, we address (a). Assume $\deg(x) = 1$, and $(\forall\alpha \lhd o(x))\mathsf{Prv}_1(x[\alpha])$, and aim for $\mathsf{Prv}_1(x)$. For that, we further assume $\mathsf{Prv}_0(y)$, and verify that for $z := x \circ y$, $\mathsf{Prv}_0(z)$, i.e. that

$$(*) \qquad \breve{\mathsf{T}}_z \wedge \mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta) \to \mathsf{Wo}_\lhd(g_{z^h}(\beta)).$$

For each $\alpha \lhd o(x)$, $\mathsf{Prv}_1(x[\alpha])$ and $\mathsf{Prv}_0(y)$ yield $\mathsf{Prv}_0(x[\alpha] \circ y)$, that is, $\mathsf{Prv}_0(z[\alpha])$. Now $\breve{\mathsf{T}}_z$ and (c) yields $\mathsf{Prog}_\lhd(\mathcal{C}_z)$, so $\mathsf{Wo}_\lhd(\beta)$ and $\mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$ imply $\beta \in \mathcal{C}_z$, thus the definition of $\mathcal{C}_z$ says $\mathsf{Wo}_\lhd(g_{z^h}(\beta))$. This concludes the verification of $\mathsf{Prv}_0(z)$.

Eventually, we look at (b). Similar to (a), assume $\deg(x) = 2$ and $\forall\alpha\mathsf{Prv}_1(x(\alpha))$, and aim for $\mathsf{Prv}_1(x)$. For that, we further assume $\mathsf{Prv}_0(y)$, and again verify $(*)$ for $z := x \circ y$. Thereto we fix a set Y and check that $\mathsf{TI}_\lhd(Y, g_{z^h}(\beta))$ follows from $\breve{\mathsf{T}}_x \wedge \mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$ (say $\beta \geq \omega$, so $1{+}\beta = \beta$). Since $\mathsf{Prog}_\lhd(\mathcal{C}_z)$ follows from $\breve{\mathsf{T}}_z$ (cf. Lemma II.4.9), $\mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$ yields $\beta{+}1 \in \mathcal{C}_z$, so $\breve{\mathsf{T}}_{z(\beta+1)}$, which implies $\mathsf{p}_1(\breve{\mathsf{T}}_{z(\beta)})$ (cf. Lemma II.4.4 (ii)). Further, $\breve{\mathsf{T}}_{z(\beta+1)}$ entails $\mathsf{Wo}_\lhd(\beta)$. Working in a model X of $\breve{\mathsf{T}}_{z(\beta)}$ that contains Y, we also have $\mathsf{Wo}_\lhd(\beta){\restriction}X$ and $\mathsf{TI}_\lhd(\mathcal{C}_{z(\beta)}{\restriction}X, \beta)$ (as $\mathcal{C}_{z(\beta)}{\restriction}X$ is a set). By assumption we have $\mathsf{Prv}_1(x(\beta))$ and $\mathsf{Prv}_0(y)$, thus also $\mathsf{Prv}_0(z(\beta))$. Actually, we want $\mathsf{prv}_0(z(\beta))$, as we need $\mathsf{Prv}_0(z(\beta)){\restriction}X$. However, this is indeed what we get, if we take care of the distinction of Prv_m and prv_m. Summing up, the model X satisfies $\breve{\mathsf{T}}_{z(\beta)} \wedge \mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$ and $\mathsf{Prv}_0(z(\beta))$, which yields $\mathsf{Wo}^X_\lhd(g_{(z(\beta))^h}(0))$ by the definition of $\mathsf{Prv}_0(z(\beta))$. As $Y \dot{\in} X$, we also have $\mathsf{TI}_\lhd(Y, g_{(z(\beta))^h}(0))$, and as $g_{z^h}(\beta) = g_{z^h[\beta]}(0) \trianglelefteq g_{(z(\beta))^h}(0)$ (cf. Lemma II.5.10 (ii)), $\mathsf{TI}_\lhd(Y, g_{z^h}(\beta))$ follows. This concludes the verification of $\mathsf{Prv}_0(z)$.

II.5.4 Proof of the main result (for the case $N_0 = 2$)

Eventually, we can observe our modular approach at work. First, we show that $\breve{\mathsf{T}}$ proves $\mathsf{it}(g)$ (i.e. that $\breve{\mathsf{T}}_{q_0}$ proves $g_{q_0^h}$).

Lemma II.5.11. $\mathsf{ACA}_0 \vdash \mathsf{Prv}_0(q_0)$.

Proof Assume (ACA), $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{TI}_\lhd(\mathcal{C})$ for $\mathcal{C} := \{\xi : \mathsf{Wo}_\lhd(f(\xi))\}$ and $f := \mathsf{it}(g)$. It suffices to show that $\mathsf{Prog}_\lhd(\mathcal{C})$. Clearly, $0 \in \mathcal{C}$. Assume $\alpha \in \mathcal{C}$ and $\mathsf{Prog}_\lhd(Y)$. Then, by Gentzen's observation, $\mathsf{Prog}_\lhd(Y^*)$ for $Y^* := \{\beta : (\forall\xi \subseteq Y)(\xi{+}\omega^\beta \subseteq Y)\}$. With $\alpha \in \mathcal{C}$, $f(\alpha) \subseteq Y^*$, so $f(\alpha){+}1 \in Y^*$ and $Y \supseteq \omega^{f(\alpha)+1} = g(f(\alpha){+}1) \trianglerighteq f(\alpha{+}1)$ (as $f = \mathsf{it}(g)$ and by Lemma I.3.9). The limit case is by the continuity of f. $\square$

The next has auxiliary character. It states that if T_x proves g_{x^h}, then $\breve{\mathsf{T}}_{x+1}$ and $\mathsf{Wo}_\lhd(\alpha)$ implies already $\mathsf{Wo}_\lhd(g_{x^h}(\alpha))$. With enough transfinite induction at hand, this allows to jump from α to $\mathsf{it}(g_{x^h}, \alpha) = g_{x^h+1}(\alpha)$.

Lemma II.5.12. $\mathsf{ACA}_0 \vdash (\forall x \in Q_2)[\breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \wedge \mathsf{Wo}_\lhd(\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))]$.

Proof Suppose $\breve{\mathsf{T}}_{x+1}$, $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{prv}_0(x)$. Further, we fix a set Y that is progressive w.r.t. $\lhd$, and aim to show that $g_{x^h}(\alpha) \subseteq Y$. As $\breve{\mathsf{T}}_{x+1}$ iff $\mathsf{p}_1(\breve{\mathsf{T}}_x)$, there is an X with $Y \dot\in X$ and $\breve{\mathsf{T}}_x{\restriction}X$. Further, $\mathcal{C}_x{\restriction}X$ is a set. Hence $\mathsf{Wo}_\lhd(\alpha)$ implies $\mathsf{TI}_\lhd(\mathcal{C}_x{\restriction}X, \alpha)$. Now $\breve{\mathsf{T}}_x{\restriction}X \wedge \mathsf{TI}_\lhd(\mathcal{C}_x{\restriction}X, \alpha)$ together with $\mathsf{prv}_0(x)$ implies $\mathsf{Wo}^X_\lhd(g_{x^h}(\alpha))$. As $Y \dot\in X$, this yields $\mathsf{TI}_\lhd(Y, g_{x^h}(\alpha))$, and $g_{x^h}(\alpha) \subseteq Y$ is by $\mathsf{Prog}_\lhd(Y)$. □

The next lemma is a refined variant of (c) of the sketch in the previous subsection. If $o(x)$ is a limit of the form $\delta_0{+}\gamma$, then the assumption is slightly weakened to $(\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\delta_0{+}\alpha])$.

Lemma II.5.13. ACA_0 *proves the following.*

(i) $x \in Q_2 \wedge \breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \to \mathsf{Prog}_\lhd(\mathcal{C}_{x+1})$.

(ii) $x \in Q_2^* \wedge \deg(x) = 1 \wedge o(x) = \delta_0{+}\gamma \wedge (\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\delta_0{+}\alpha]) \wedge \breve{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\mathcal{C}_x)$.

Proof (i) We assume $\breve{\mathsf{T}}_{x+1}$ and $\mathsf{prv}_0(x)$, and aim to show $\mathsf{Prog}_\lhd(\mathcal{C}_{x+1})$, that is, $\mathsf{Prog}_\lhd(\{\alpha : \mathsf{Wo}_\lhd(\mathsf{it}(g_{x^h}, \alpha))\})$. If $\gamma \subseteq \mathcal{C}_{x+1}$, then $\gamma \in \mathcal{C}_{x+1}$ as $\mathsf{it}(g_{x^h})$ is continuous. Next assume that $\alpha \in \mathcal{C}_{x+1}$, i.e. $\mathsf{Wo}_\lhd(\mathsf{it}(g_{x^h}, \alpha))$. Then also $\mathsf{Wo}_\lhd(\mathsf{it}(g_{x^h}, \alpha){+}1)$. By Lemma II.5.12 we have $\forall\alpha[\mathsf{Wo}_\lhd(\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))]$. Therefore, $\mathsf{Wo}_\lhd(g_{x^h}(\beta))$ for $\beta := \mathsf{it}(g_{x^h}, \alpha){+}1$ follows. As $\mathsf{it}(g_{x^h}, \alpha{+}1) \unlhd g_{x^h}(\mathsf{it}(g_{x^h}, \alpha){+}1)$ (cf. Lemma I.3.9), $\alpha{+}1 \in \mathcal{C}_{x+1}$ follows.

(ii) Assume $(\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\delta_0{+}\alpha])$, $\breve{\mathsf{T}}_x$, $\deg(x) = 1$ and $o(x) = \delta_0{+}\gamma$. Hence $x =_{NF} y \circ z$ where either $y = (\gamma, v)$ or $y = (1, (\gamma, q_0)^-)$, so $o(x^h) = o(x^H) = o(x)$. We aim for $\mathsf{Prog}_\lhd(\mathcal{C})$, where $\mathcal{C} := \mathcal{C}_x = \{\alpha : \mathsf{Wo}_\lhd(g_{x^h}(\alpha))\}$. Since g_{x^h} is normal, it suffices to show that $\alpha \in \mathcal{C}$ implies $\alpha{+}1 \in \mathcal{C}$, and that $0 \in \mathcal{C}$. Assume that $\alpha \in \mathcal{C}$. Let $(s_\xi : \xi \lhd \gamma)$ be a sequence with $\mathsf{sup}_{\xi\lhd\gamma} s_\xi = g_{x^h}(\alpha{+}1)$ as in Lemma II.3.5 (iii), i.e. $s_0 := g_{x^h}(\alpha){+}1$, $s_{\xi+1} := g_{x^h[\delta_0+\xi]}(s_\xi)$ and $s_{\gamma'} := \mathsf{sup}_{\xi\lhd\gamma'} s_\xi$ (to show that $0 \in \mathcal{C}$, start with $s_0 := 0$ and proceed similar as below). Now $\alpha{+}1 \in \mathcal{C}$ follows if $\mathsf{Wo}_\lhd(s_\xi)$ for each $\xi \lhd \gamma$. To show this, fix a $\xi_0 \lhd \gamma$ and a progressive set Y. As $\breve{\mathsf{T}}_x$ entails $\mathsf{p}_1(\breve{\mathsf{T}}_{x[\delta_0+\xi_0+1]})$ by definition of $\breve{\mathsf{T}}_x$, there is an X so that $Y \dot\in X$ and $\breve{\mathsf{T}}_{x[\delta_0+\xi_0+1]}{\restriction}X$, so also $\breve{\mathsf{T}}_{x[\delta_0+\xi_0]+1}{\restriction}X$ by Lemma II.4.4 (ii). Next, we show that

$$(*) \qquad C := \{\xi \unlhd \xi_0 : \mathsf{Wo}^X_\lhd(s_\xi)\} \text{ is progressive w.r.t. } \lhd\,.$$

Since C is a set and $\breve{\mathsf{T}}_x$ implies $\mathsf{Wo}_{\leadsto^*}(x)$ which in turn yields $\mathsf{Wo}_\lhd(\gamma)$, $(*)$ implies $\mathsf{TI}_\lhd(Y, s_{\xi_0})$. Since Y was arbitrary, we have $\mathsf{Wo}_\lhd(s_{\xi_0})$, therefore $\mathsf{Prog}_\lhd(\mathcal{C})$.

For $(*)$, we again just show the successor case (the limit case is by the continuity of $\eta \mapsto s_\eta$). Let $\xi \in C$, so $0 \lhd \xi \unlhd \xi_0 \lhd \gamma$. By the premise of (ii) we have $\mathsf{prv}_0(x[\delta_0{+}\xi])$, Now $\breve{\mathsf{T}}_{x[\delta_0+\xi]+1}{\upharpoonright}X$ and the "small variant" of Lemma II.5.12 yields $\forall\alpha[\mathsf{Wo}^X_\lhd(\alpha) \to \mathsf{Wo}^X_\lhd(g_{(x[\delta_0+\xi])^h}(\alpha))]$. Since by Lemma II.5.10 (i), $s_{\xi+1} = g_{x^h[\delta_0+\xi]}(s_\xi) \unlhd g_{(x[\delta_0+\xi])^h}(s_\xi)$, $\mathsf{Wo}^X_\lhd(s_\xi)$ implies $\mathsf{Wo}^X_\lhd(s_{\xi+1})$, i.e. $\xi{+}1 \in C$. $\square$

As a consequence of Lemma II.5.13 (i), we obtain that p_1 proves it.

Lemma II.5.14. $\mathsf{ACA}_0 \vdash \mathsf{Prv}_1(q_1)$.

Proof Let $x \in Q_2$, assume $\mathsf{prv}_0(x)$ and aim for $\mathsf{Prv}_0(x{+}1)$. For that, further assume $\breve{\mathsf{T}}_{x+1}$, $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{TI}_\lhd(\mathcal{C}_{x+1}, \alpha)$. Now $\breve{\mathsf{T}}_{x+1}$ and $\mathsf{prv}_0(x)$ yield $\mathsf{Prog}_\lhd(\mathcal{C}_{x+1})$ by Lemma II.5.13 (i). Together with $\mathsf{TI}_\lhd(\mathcal{C}_{x+1}, \alpha)$ we conclude $\alpha \in \mathcal{C}_{x+1}$, which says $\mathsf{Wo}_\lhd(g_{x^h+1}(\alpha))$. $\square$

The next lemma corresponds to (b) of the sketch and shows that for names of degree two, the approximation $x(\alpha)$ goes well with $\mathsf{Prv}_1(x)$. Note that the proof only looks simpler than in the sketch, as part of the proof is hidden in the proof of Lemma II.5.12.

Lemma II.5.15. $\mathsf{ACA}_0 \vdash (\forall x \in Q_2^*)[\deg(x) = 2 \wedge \forall\alpha \mathsf{prv}_1(x(\alpha)) \to \mathsf{Prv}_1(x)]$.

Proof Assume that $x \in Q_2^*$ with $\deg(x) = 2$, and $\forall\alpha\mathsf{prv}_1(x(\alpha))$, and aim for $\mathsf{Prv}_1(x)$. For that, further assume that $\mathsf{prv}_0(y)$ and $z := x \circ y \in Q_2$, and aim for $\mathsf{Prv}_0(z)$. To verify $\mathsf{Prv}_0(x \circ y)$, note that $(x \circ y)(\alpha) = x(\alpha) \circ y$ (cf. Lemma II.2.12), and suppose

$$\breve{\mathsf{T}}_z \wedge \mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta).$$

We have to show $\mathsf{Wo}_\lhd(g_{z^h}(\beta))$. $\breve{\mathsf{T}}_z$ and Lemma II.4.9 yield $\mathsf{Prog}_\lhd(\mathcal{C}_z)$. Together with $\mathsf{Wo}_\lhd(\beta)$ and $\mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$, we obtain $\breve{\mathsf{T}}_{z(\beta'+1)}$, where $\beta' := 1{+}\beta$, which by Lemma II.4.4 (ii) gives $\breve{\mathsf{T}}_{z(\beta')+1}$. By assumption, we have $\mathsf{prv}_1(x(\beta'))$ and $\mathsf{prv}_0(y)$, thus $\mathsf{prv}_0(x(\beta') \circ y)$, that is, $\mathsf{prv}_0(z(\beta'))$. Since trivially $\mathsf{Wo}_\lhd(0)$, $\breve{\mathsf{T}}_{z(\beta')+1}$ and $\mathsf{prv}_0(z(\beta'))$ yield $\mathsf{Wo}_\lhd(g_{(z(\beta'))^h}(0)$ by Lemma II.5.12. As $g_{z^h}(\beta) = g_{z^h[1+\beta]}(0) = g_{z^h[\beta']}(0) \unlhd g_{(z(\beta'))^h}(0)$ (cf. Lemma II.5.10 (ii)), $\mathsf{Wo}_\lhd(g_{z^h}(\beta))$ follows. $\square$

The first claim of the next Lemma corresponds to (a) of the sketch, and the second is an immediate consequence, which readily entails $\mathsf{Prv}_2(q_1)$ (cf. Lemma II.5.18).

Lemma II.5.16. ACA_0 *proves the following: for each* $x \in Q_2^*$ *with* $\deg(x) = 1$ *and* $o(x) = \delta_0{+}\gamma$, *and each* $(1, v) \in Q_2^*$,

(i) $(\forall\alpha \lhd \gamma)\mathsf{prv}_1(x[\delta_0{+}\alpha]) \to \mathsf{Prv}_1(x)$.

(ii) $\mathsf{Prv}_1(1, v) \wedge \mathsf{prv}_1(1, v) \to \forall\alpha\mathsf{Prv}_1(1{+}\alpha, v)$.

Proof (i) Assume $x \in Q_2^*$ with $\deg(x) = 1$, $o(x) = \delta_0 + \gamma$, and $(\forall \alpha \lhd \gamma)\mathsf{prv}_1(x[\delta_0 + \alpha])$, and aim for $\mathsf{Prv}_1(x)$. For that, further assume that $\mathsf{prv}_0(y)$ and $z := x \circ y \in Q_2^*$, and aim for $\mathsf{Prv}_0(z)$. Thereto, let δ_1 so that for each β, $x[\beta] \circ y = z[\delta_1 + \beta]$, and so $o(z) = \delta_1 + \delta_0 + \gamma$ (cf. Lemma II.2.9). To verify $\mathsf{Prv}_0(z)$, we suppose that

$$\breve{\mathsf{T}}_z \wedge \mathsf{Wo}_\lhd(\beta) \wedge \mathsf{TI}_\lhd(\mathcal{C}_z, \beta).$$

We show $\mathsf{Wo}_\lhd(g_{z^h}(\beta))$. First, note that for $\alpha \lhd \gamma$, $\mathsf{prv}_1(x[\delta_0 + \alpha])$ and $\mathsf{prv}_0(y)$ yield $\mathsf{prv}_0(x[\delta_0 + \alpha] \circ y)$, that is, $\mathsf{prv}_0(z[\delta_1 + \delta_0 + \alpha])$. Now $\breve{\mathsf{T}}_z$ and $(\forall \alpha \lhd \gamma)\mathsf{prv}_0(z[\delta_1 + \delta_0 + \alpha])$ yield $\mathsf{Prog}_\lhd(\mathcal{C}_z)$ by Lemma II.5.13 (ii). Finally, $\mathsf{Wo}_\lhd(\beta)$ and $\mathsf{TI}_\lhd(\mathcal{C}_z, \beta)$ imply $\beta \in \mathcal{C}_z$. The definition of $\mathcal{C}_z$ yields $\mathsf{Wo}_\lhd(g_{z^h}(\beta))$. This concludes the verification of $\mathsf{Prv}_0(x)$.
(ii) By induction on α (in the sense of Corollary I.4.3). For $\alpha = 0$, the claim holds by assumption. For the successor case, note that $\mathsf{Prv}_1(1, v)$ and $\mathsf{prv}_1(1+\alpha, v)$ yield $\mathsf{prv}_1(1+\alpha+1, v)$. In the limit case, we need to show that $(\forall \alpha \lhd \gamma)\mathsf{prv}_1(1+\alpha, v)$ implies $\mathsf{Prv}_1(\gamma, v)$, which is by (i). □

The next Lemma elaborates on the argument from the sketch that $\forall \beta \mathsf{Prv}_2(1+\beta, q_0)$ follows from $\mathsf{Prv}_2(1, q_0)$ by induction on β. The first claim of the lemma addresses the limit case. Observe that the Lemma is a special case ($v = q_0$) of the one-up variant of II.5.16 where Prv_1 and prv_1 are replaced by Prv_2 and prv_2, respectively.

Lemma II.5.17. ACA_0 *proves the following:*

(i) $(\forall \alpha \lhd \gamma)\mathsf{prv}_2(1+\alpha, q_0) \to \mathsf{Prv}_2(\gamma, q_0)$.

(ii) $\mathsf{Prv}_2(1, q_0) \wedge \mathsf{prv}_2(1, q_0) \to \forall \alpha \mathsf{Prv}_2(1+\alpha, q_0)$.

Proof Assumed that $(\forall \beta \lhd \gamma)\mathsf{Prv}_2(1+\beta, q_0)$, and aim for $\mathsf{Prv}_2(\gamma, q_0)$. Thereto, further assume that $\mathsf{prv}_1(1, y)$, and verify $\mathsf{Prv}_1(1, z)$ for $z := (\gamma, q_0) \circ y$. Observe that $\deg(1, z) = 2$, $\deg(1, z^-) = 1$ and $(1, z)[\alpha] = (1, z^-)[\alpha]$. Further, note that the assumptions $(\forall \beta \lhd \gamma)\mathsf{Prv}_2(1+\beta, q_0)$ and $\mathsf{prv}_1(1, y)$ yield $(\forall \beta \lhd \gamma)\mathsf{Prv}_1((1, z^-)[\beta])$. Hence Lemma II.5.16 (i) yields $\mathsf{Prv}_1(1, z^-)$, and its "small variant" yields $\mathsf{prv}_1(1, z^-)$. Now Lemma II.5.16 (ii) implies $\forall \beta \mathsf{prv}_1(1+\beta, z^-)$, that is $\forall \beta \mathsf{prv}_1((1, z)(\beta))$. Finally $\mathsf{Prv}_1(1, z)$ is by Lemma II.5.15.
(ii) Again, this is shown using Theorem I.4.2: the case $\alpha = 0$ and the successor case are shown as in the proof of Lemma II.5.16 (ii), and the limit case is by (i). □

Lemma II.5.18. $\mathsf{ACA}_0 \vdash \mathsf{Prv}_2(q_1)$

Proof To show $\mathsf{Prv}_2(q_1)$, assume that $y \in Q_1$ and $\mathsf{prv}_1(1, y)$, and aim for $\mathsf{Prv}_1(x)$ for $x := (1, y+1)$. Note that $\deg(x) = 2$. Once we know that $\forall \alpha \mathsf{prv}_1(x(\alpha))$, the claim is by Lemma II.5.15. By the small variant of Lemma II.5.16 (ii), $\mathsf{prv}_1(1, y)$ yields $\forall \alpha \mathsf{prv}_1(1+\alpha, y)$. Further, $\mathsf{prv}_1(1+\alpha, y)$ and $\mathsf{prv}_1(q_1)$ yield $\mathsf{prv}_1((1+\alpha, y)+1)$, that is, $\mathsf{prv}_1(x(\alpha))$. Thus indeed $\forall \alpha \mathsf{prv}_1(x(\alpha))$. □

Theorem II.5.19. $\mathsf{T}^\epsilon \vdash (\forall x \in Q_1^*)\mathsf{Prv}_2(x)$ *and* $\mathsf{T}^\epsilon \vdash (\forall x \in Q_2^*)\mathsf{Prv}_1(x)$.

Proof The first claim is immediate Lemma II.5.18 and Lemma II.5.17 (ii) (recall that with $\mathsf{ACA}_0 \vdash \mathsf{Prv}_2(q_1)$ also $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_2(q_1)$, hence $\mathsf{T}^\epsilon \vdash \mathsf{prv}_2(q_1)$). The second claim is shown by induction on $\rightsquigarrow^*$ (in the sense of Theorem I.4.2). We consider the following possible cases.

(i) $x = y{+}1$. If $x = q_1$, this is Lemma II.5.14. Else, we have $\mathsf{prv}_1(y)$ by I.H. Together with $\mathsf{Prv}_1(q_1)$, this yields $\mathsf{Prv}_1(x)$.

(ii) $\deg(x) = 1$ and $o(x) = \gamma$. Then by I.H., $(\forall \alpha \lhd \gamma)\mathsf{prv}_1(x[\alpha])$, and the claim is by Lemma II.5.16 (i).

(iii) $\deg(x) = 2$. In this case, $x =_{NF} (1, (1{+}\beta, q_0)) \circ y$. Then $y \rightsquigarrow^* x$ (cf. Lemma II.2.16 (iv)), thus by I.H., $\mathsf{prv}_1(y)$. Further, we have $\mathsf{Prv}_2(1{+}\beta, q_0)$ by the first claim, which together with $\mathsf{prv}_1(1, q_0)$ yields $\mathsf{Prv}_1(1, (1{+}\beta, q_0))$. Now, $\mathsf{Prv}_1(1, (1{+}\beta, q_0))$ and $\mathsf{prv}_1(y)$ yield $\mathsf{Prv}_1(x)$.

□

Corollary II.5.20. $\mathsf{T}^\epsilon \vdash (\forall x \in Q_2)\mathsf{Prv}_0(x)$.

Proof $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(q_0)$ is by Lemma II.5.11, and if $x \in Q_2^*$, then $\mathsf{Prv}_1(x)$ is by Theorem II.5.19, which together with $\mathsf{prv}_0(q_0)$ yields $\mathsf{Prv}_0(x)$. □

The corollary immediately provides lower bounds for the proof-theoretic ordinal of a theory of the form T_x or $\mathsf{T}_x + (\mathsf{I_N})$ (formula induction, see page 18). Having $(\mathsf{I_N})$ at hand, $\mathsf{TI}_\lhd(\mathcal{C}_x, \alpha)$ is provable for each $x \in Q_{N_0}$ and each $\alpha < \varepsilon_0$. Thus, if T_x proves g_{x^h}, then for each $\alpha < \varepsilon_0$, $\mathsf{T}_x \vdash \mathsf{Wo}_\lhd(g_{x^h}(\alpha))$.

Below, we list a few instances of the above corollary. The presentation of the ordinals in the form $\varphi\vec{\alpha}$ is due to Definition IV.5.14 and Corollary IV.5.16. Also recall (see Example II.0.8) that $\mathsf{p}_2((\mathsf{ACA}))$ is $(\Sigma^1_1\text{-}\mathsf{DC})$, $\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ is (ATR) and $\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA}))$ is $\mathsf{p}_1\mathsf{p}_2((\mathsf{ACA})) \wedge (\Sigma^1_1\text{-}\mathsf{DC})$.

Example II.5.21.

(i) $|\mathsf{ACA}_0| \geq g_{q_0^h}(\omega) = g_{q_1}(\omega) = \varphi 10 = \varepsilon_0$.

(ii) $|\mathsf{ACA}_0 + (\mathsf{I_N})| \geq g_{q_1}(\varepsilon_0) = \varphi 1 \varepsilon_0$.

(iii) $|\mathsf{p}_1(\mathsf{ACA}_0)| \geq g_{q_1^h}(\omega) = g_{(2,q_0)}(\omega) = \varphi 20$.

(iv) $|\mathsf{p}_1(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{(2,q_0)}(\varepsilon_0) = \varphi 2 \varepsilon_0$.

(v) $|\mathsf{p}_2(\mathsf{ACA}_0)| \geq g_{q_2^h}(\omega) = g_{q_2}(\omega) = g_{q_2[\omega]}(0) = \varphi \omega 0$.

(vi) $|\mathsf{p}_2(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{q_2}(\varepsilon_0) = g_{q_2[\varepsilon_0]}(0) = \varphi\varepsilon_0 0$.

(vii) $|\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0)| \geq g_{q_2+1}(\omega) = \varphi 100 = \Gamma_0$ *(Feferman-Schütte ordinal).*

(viii) $|\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{q_2+1}(\varepsilon_0) = \varphi 10\varepsilon_0$.

(ix) $|\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0)| \geq g_{(2,q_1)}(\omega) = \varphi 1\omega 0$.

(x) $|\mathsf{p}_2\mathsf{p}_1\mathsf{p}_2(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{(2,q_1)}(\varepsilon_0) = \varphi 1\varepsilon_0 0$.

(xi) $|\mathsf{p}_1\mathsf{p}_2^2(\mathsf{ACA}_0)| \geq g_{(1,q_1^2)+1}(\omega) = \varphi 1000$ *(Ackermann ordinal).*

(xii) $|\mathsf{p}_2^{n+2}(\mathsf{ACA}_0)| \geq g_{(1,q_1^{n+2})[\omega]}(0) = g_{(\omega,q_1^{n+1})} = \varphi\omega\underbrace{0\ldots0}_{n}0$.

(xiii) $|\mathsf{p}_2^{n+2}(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{(\varepsilon_0,q_1^{n+1})}(0) = \varphi\varepsilon_0\underbrace{0\ldots0}_{n}0$,

(xiv) $|\mathsf{p}_1\mathsf{p}_2^{n+2}(\mathsf{ACA}_0)| \geq g_{(1,q_1^{n+2})+1}(\omega) = \varphi 1\underbrace{0\ldots0}_{n+1}0$.

(xv) $|\mathsf{p}_1\mathsf{p}_2^{n+2}(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq g_{(1,q_1^{n+2})+1}(\varepsilon_0) = \varphi 1\underbrace{0\ldots0}_{n+1}\varepsilon_0$.

(xvi) $|\bigcup_n \mathsf{p}_2^{n+1}(\mathsf{ACA}_0)| \geq \mathsf{sup}_n \varphi 1\underbrace{0\ldots0}_{n}0$ *(small Veblen number).*

A tiny extension of the above procedure gives now a lower bound for $\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA}_0)$. We pick q_3 as a name for $\mathsf{p}_3\mathsf{p}_1$, and let

$$\mathsf{Prv}_0(q_3) := \breve{\mathsf{T}}_{q_3} \wedge \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_{q_3},\alpha) \to \mathsf{Wo}_\lhd(g_{q_3}(\alpha))],$$

where $g_{q_3} := [\mathsf{It}_3, \mathsf{It}, \mathsf{it}, g]$ and $\mathcal{C} := \{\alpha : \mathsf{p}_2^{1+\alpha}\mathsf{p}_1((\mathsf{ACA}))\}$. Exactly as in the proof of Lemma II.4.9, one shows that $\mathsf{p}_3\mathsf{p}_1(\mathsf{ACA}_0) \vdash \mathsf{Prog}_\lhd(\mathcal{C}_{q_3})$. Then, $\mathsf{ACA}_0 \vdash \mathsf{Prv}_0(q_3)$ is easily obtained: Assume that $\mathsf{p}_3\mathsf{p}_1((\mathsf{ACA}))$, $\mathsf{Wo}_\lhd(\alpha)$ and $\mathsf{TI}_\lhd(\mathcal{C}_{q_3},\alpha)$, and aim for $\mathsf{Wo}_\lhd(g_{q_3}(\alpha))$. For that, pick an $\lhd$-progressive set Y. $\mathsf{p}_3\mathsf{p}_1((\mathsf{ACA}))$ implies $\mathsf{Prog}_\lhd(\mathcal{C}_{q_3})$, hence $2{+}\alpha \in \mathcal{C}_{q_3}$, that is $\mathsf{p}_2^{2+\alpha}\mathsf{p}_1((\mathsf{ACA}))$, which is Π^1_3. Since Π^1_3-reflection is at hand, there is an ω-model X of ACA_0 with $Y \dot{\in} X$ and $\mathsf{p}_2^{2+\alpha}\mathsf{p}_1((\mathsf{ACA}))\restriction X$. As $x := (1,(2{+}\alpha,q_0)) \in Q_2^*$, Theorem II.5.19 yields $\mathsf{prv}_1(x)$. Further, $\mathcal{C}_x\restriction X$ is a set. Therefore, $\mathsf{Wo}_\lhd^X(g_{x^h}(0))$, in particular, $\mathsf{Wo}_\lhd^X(\mathsf{It}^{2+\alpha}[\mathsf{it},g,0])$, that is, $\mathsf{TI}_\lhd(Y, g_{q_3}(\alpha))$. As Y was arbitrary, $\mathsf{Wo}_\lhd(g_{q_3}(\alpha))$ follows.

The proof of $\mathsf{Prv}_1(q_1)$ immediately yields that $\mathsf{p}_1\mathsf{p}_3\mathsf{p}_1((\mathsf{ACA}))$ proves $\mathsf{it}(g_{q_3})$, and therefore, $\mathsf{p}_1\mathsf{p}_3\mathsf{p}_1((\mathsf{ACA})) \vdash \mathsf{Wo}_\lhd(\mathsf{it}[g_{q_3},n])$ for each n, so $|\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA}_0)| \geq \mathsf{it}(g_{q_3},\omega)$. We will see that this bound is indeed sharp, and that $\mathsf{it}[g_{q_3},\omega] = g'_{q_3}(0) = \vartheta\Omega^\Omega$ (big Veblen number).

Chapter III

The general case

This chapter extends what we have done in the previous one to the general case, where now all operations and functionals are considered which are built by iterated transfinite composition from the basic functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$ and the basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$. All these operations play a role in the reduction process of $\mathsf{p}_{n+1}(\mathsf{ACA}_0)$.

In order to have enough names to address these operations and functionals, we first extend the ordered sets $(Q_2^H, <_2^H)$ and $(Q_2, <_2)$ to $(Q^H, <^H)$ and $(Q, <)$, respectively. Then, all the related concepts are generalized so that its relevant properties are preserved, in particular, we still have that if $\deg(x) = m{+}1$

$$\mathsf{Op}_x \Leftrightarrow (\forall \alpha \lhd o(x))(\mathsf{p}_{m+1} \circ \widehat{\mathsf{Op}}_{x[\alpha]}).$$

To state and prove the main result of this chapter, we also have to lift the notion of Op_y proves H_{y^H} to higher types. Recall that we said that Op_x^+ proves $H_{x^*}^+$, if whenever $\mathsf{Op}_{(1,y)}$ proves $H_{(1,y)^H}$, then $\mathsf{Op}_x^+(\mathsf{Op}_{(1,y)})$ proves $H_{x^*}^+(H_{(1,y)^H})$. Accordingly, we say that $\mathsf{Op}_x^{+(m+1)}$ proves $H_{x^*}^{+(m+1)}$, if for each $i \le m$, $\mathsf{Op}_{(1,y_i)}^{+i}$ proves $H_{(1,y_i)^H}^{+i}$, then

$$\mathsf{Op}_x^{+(m+1)} \circ \mathsf{Op}_{(1,y_m)}^{+m} \circ \ldots \circ \mathsf{Op}_{(1,y_0)}^{+0} \text{ proves } (H_{x^*}^{+(m+1)}, H_{(1,y_m)^H}^{+m}, \ldots, H_{(1,y_0)^H}^{+0}),$$

where Op^{+m} and H^{+m} are essentially obtained from Op and H by replacing each p_{n+1} and each It_{n+1} by p_{m+n+1} and It_{m+n+1}, respectively. We use $\mathsf{Prv}_{m+2}(x)$ to formalize the statement "$\mathsf{Op}_x^{+(m+1)}$ proves $H_{x^*}^{+(m+1)}$".

The main result of the previous chapter can be summarized as $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(q_0)$ and $\mathsf{T}^\epsilon \vdash \bigwedge_{m\le 2}(\forall x \in Q_2^*)\mathsf{Prv}_m(x)$. Now, we show that $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(q_0)$, and for each N_0,

$$\mathsf{T}^\epsilon \vdash \bigwedge_{m\le N_0} (\forall x \in Q_{N_0}^*)\mathsf{Prv}_m(x).$$

Since this is true for each N_0, we have in particular, that for each $n \in \mathbb{N}$, p_{n+1} proves $(\mathsf{It}_{n+1}, \ldots, \mathsf{It}_1)$, and so $|\mathsf{p}_{n+1}(\mathsf{ACA}_0)| \geq (\mathsf{It}_{n+1}, \ldots, \mathsf{It}_1, g, \omega)$ (where $g(\alpha) := \omega^{1+\alpha}$). The bound N_0 is given from outside, since there is no $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ so that for all n, $\mathsf{Op}^{\vartheta}_{n+1} \Leftrightarrow \mathsf{p}_{n+1}$ (there is only an $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ so that for each N_0, $(\forall n < N_0)(\mathsf{Op}^{\vartheta}_{n+1} \Leftrightarrow \mathsf{p}_{n+1})$). Hence all the following is relative to some arbitrary large but fixed $N_0 \in \mathbb{N}$, as we have to restrict to operations and functionals with names in Q_{N_0} when it comes to proofs within ACA_0 or T^{ϵ}.

This chapter is structured analogously to the previous one, and again it is assumed that we know how to translate ordinals of the form $g_x(\alpha)$ into ordinal notations (cf. Chapter IV). However, instead of discussing differences between operations and functionals, we start with an outline of how the various concepts extend to the general case.

How to extend things

We first say how we extend the name structure for operations and functionals. Then, we provide examples that illustrate relevant points. As we aim to draw the general picture, some details are still suppressed.

The extension of $(Q^H_2, <_2)$ to $(Q^H, <)$ is rather straightforward. Instead of names of level at most two, we now consider names of all finite level. So suppose that we have already defined names of level n and know how to compare them. Then, if $x_1 < \ldots < x_k$ are names of level n, $\langle(\alpha_1, x_1), \ldots, (\alpha_k, x_k)\rangle$ is a name of level n+1. Further, we assign to each n and each $x \in Q^H$ a type-n+2 functional H^{+n}_x. For each n, each $k > 1$ and each $\alpha > 0$, we set

(i) $H^{+n}_{(\alpha,q_0)} := \mathsf{It}^{\alpha}_{n+1}$,

(ii) $H^{+n}_{(\alpha,x)} := (H^{+(n+1)}_x(\mathsf{It}_{n+1}))^{\alpha}$, and

(iii) if $\langle x_1, \ldots, x_k\rangle$ is a name of level n, then $H^{+n}_{\langle x_1,\ldots,x_k\rangle} := H^{+n}_{x_1} \circ \ldots \circ H^{+n}_{x_k}$.

Further, $H_x := H^{+0}_x$ and $H^+_x := H^{+1}_x$.

Next, we extend $(Q_2, <_2)$ to $(Q, <)$. Note that e.g. $(1, (\gamma, q_0)^-) \in Q_2 \setminus Q^H_2$. We refer to $(\gamma, q_0)^-$ as a prename of level 1, and let $P_1 := \{(\gamma, q_0)^- : \gamma \in \mathsf{Lim}(\Omega)\}$. Names of level n+1 are now build form names and prenames of level n as described below.

Suppose that we have already defined names and prenames of level n and know how to compare them. Then, if $v_1 < \ldots < v_k$ are names or prenames of level n, $\langle(\alpha_1, v_1), \ldots, (\alpha_k, v_k)\rangle$ is a name of level n+1. Further, if x is a name of level n with $(x)_0 = (\gamma, v)$ or $(x)_0 = (\beta{+}1, y^-)$, then x^- is a prename of level n+1. Observe that now prenames may have length bigger than one; e.g., $\langle(1,(\omega,q_0)^-),(1,(\omega,q_0))\rangle^-$ and $((\omega{+}1,(\omega,q_0)^-))^-$ are prenames of level two.

To assign operations to names, we lift $\deg(v)$, $o(v)$, $x[\alpha]$ and $\rightsquigarrow^*$ straightforwardly. We keep the definition of $\deg(v)$ and $o(v)$ literally unchanged. One way to generalize the definition of $x[\alpha]$ is the following:

(i) $q_1[\alpha] := q_0$, and if $x = (x)_0 * y$, then $x[\alpha] := (x)_0[\alpha] * y$,

(ii) $(\beta{+}1, v)[\alpha] := (1, v)[\alpha] \circ (\beta, v)$,

(iii) $(1, x{+}1)[\alpha] := (1{+}\alpha, x)$, and $(1, x)[\alpha] := (1, x[\alpha])$ if $x \neq y{+}1$,

(iv) $(\gamma, v)[\alpha] := (\alpha, v)$ if $\alpha < \gamma$, and else $(\gamma, v)[\alpha] := (\gamma, v)$,

(v) $(1, x^-)[\alpha] := (1, x[\alpha])$.

With $x[\alpha]$ at hand, the relation $\rightsquigarrow^*$ is defined as before. This allows us to keep what we called the proper definition of the operations (seemingly) unchanged (it is changed of course, since the underlying relation $\rightsquigarrow\restriction Q_{N_0}$ is a proper extension of $\rightsquigarrow\restriction Q_2$). That is, if $\deg(x) = m{+}1$, then

$$\mathsf{Op}_x^{+n} \Leftrightarrow (\forall\alpha \lhd o(x))\mathsf{p}_{m+n+1}\widehat{\mathsf{Op}}_{x[\alpha]}^{+n}.$$

where the $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ so that for all $n \leq N_0$, $\mathsf{Op}_{n+1}^{\vartheta} \Leftrightarrow \mathsf{p}_{n+1}$. Again, we have that $\breve{\mathsf{T}}_x := \widehat{\mathsf{Op}}_x(\breve{\mathsf{T}})$ is Π^1_{m+2} if $\deg(x) = m{+}1$.

As in the case $N_0 = 2$, Op_{q_1} iff p_1, $\mathsf{Op}_{(n+1,q_1)}$ iff $\mathsf{p}_2^{n+1}\mathsf{p}_1$, and also $\mathsf{Op}_{x\circ y}^{+n}$ iff $\mathsf{Op}_x^{+n}\circ\mathsf{Op}_y^{+n}$. To get an idea how the above definition works, we have a look at the operation with name $x := (\alpha{+}1,(\omega,q_0)^-)$. We have that $\deg(x) = 1$ and $o(x) = \omega$, therefore Op_x iff $\forall n(\mathsf{p}_1\mathsf{Op}_{x[n]})$. So let us figure out what $\mathsf{Op}_{x[n]}$ is. According to the above definition, we find that $x[n] = (1,(\omega,q_0)^-)[n] \circ (\alpha,(\omega,q_0)^-) = \langle(1,(n{+}1,q_0)),(\alpha,(\omega,q_0)^-)\rangle$. Hence we have that Op_x iff $\forall n(\mathsf{p}_1\mathsf{p}_2^{n+1}\mathsf{p}_1 \circ \mathsf{Op}_{(\alpha,(\omega,q_0)^-)})$ iff $\mathsf{Op}_{(1,(\omega,q_0)^-)} \circ \mathsf{Op}_{(\alpha,(\omega,q_0)^-)}$.

Further, let $x_1 := (1,(1,(\omega,q_0)))$, $x_2 := (1,(1,(\omega,q_0)^-))$ and $x_3 := (1,(1,(\omega,q_0)^-)^-)$. We have that $o(x_1) = o(x_2) = o(x_3) = \omega$, $\deg(x_1) = 3$, $\deg(x_2) = 2$, $\deg(x_3) = 1$, and

(i) Op_{x_1} iff $\mathsf{p}_3^{\omega}\mathsf{p}_2\mathsf{p}_1$ iff $\mathsf{p}_3^{\omega}\mathsf{p}_1$,

(ii) Op_{x_2} iff $\forall n(\mathsf{p}_2\mathsf{p}_3^{n}\mathsf{p}_2)\mathsf{p}_1$, and

(iii) Op_{x_3} iff $\forall n(\mathsf{p}_1\mathsf{p}_3^{n}\mathsf{p}_1)$.

As in the case $N_0 = 2$, an operation Op_x corresponds to the type-2 functional H_{x^H}, and behaves as a type-2 object, while $\mathsf{Op}_x^{+(m+1)}$ corresponds to the type-m+3 functional $H_{x^*}^{+(m+1)}$, and behaves as a type-m+3 object. Due to the higher type nature of Op_y^{+m}, Corollary II.4.7, which stated that for each open Π^1_2-sentence $\breve{\mathsf{T}}'$,

$$\mathsf{T}^\epsilon \vdash y \in Q_1^* \wedge \mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \rightarrow \mathsf{Op}_{(1,y)}(\breve{\mathsf{T}}'),$$

canonically extends to the following (cf. Corollary III.6.8): for each open Π^1_{n+2}-sentence $\breve{\mathsf{T}}'$,

$$(*) \qquad \mathsf{T}^\epsilon \vdash y \in Q^*_{N_0-n-1} \wedge \mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \rightarrow \mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}').$$

This allows us to prove that if $\deg(x) > 1$, then $\breve{\mathsf{T}} \rightarrow \mathsf{Prog}_{\lhd}\{\alpha : \breve{\mathsf{T}}_{x(\alpha)}\}$, analogously as in the previous chapter, which is a key result that helps us lift the entire proof.

Let us look at an instance of $(*)$. Thereto, observe that $\mathsf{p}_3^\omega \mathsf{p}_1$ iff $\mathsf{Op}^+_{(1,y)}\mathsf{p}_1$ for $y = (\omega, q_0)$ and that $\mathsf{p}_3^\omega \mathsf{p}_1$ implies $\forall n(\mathsf{p}_2\mathsf{p}_3^n\mathsf{p}_2)\mathsf{p}_1$, in other words, $\mathsf{Op}^+_{(1,y)}\mathsf{p}_1 \Rightarrow \mathsf{Op}^+_{(1,y)(0)}\mathsf{p}_1$ (we still have that $(1,y)(\alpha) := (1{+}\alpha, (\omega, q_0)^-))$. Further, with enough transfinite induction at hand, $\mathsf{p}_3^\omega \mathsf{p}_1$ implies $\mathsf{Op}^+_{(1,y(\alpha))}\mathsf{p}_1$, that is, $(\forall n(\mathsf{p}_2\mathsf{p}_3^n\mathsf{p}_2))^\alpha \mathsf{p}_1$: suppose we have that $\mathsf{Op}^+_{(1,y)}\mathsf{p}_1 \Rightarrow \mathsf{Op}^+_{(1,y)(\alpha)}\mathsf{p}_1$; since $\mathsf{Op}^+_{(1,y)(\alpha)}\mathsf{p}_1(\breve{\mathsf{T}})$ is Π^1_3, $\mathsf{Op}^+_{(1,y)}\mathsf{p}_1(\breve{\mathsf{T}})$ implies $(\mathsf{Op}^+_{(1,y)} \circ \mathsf{Op}^+_{(1,y)(\alpha)})\mathsf{p}_1(\breve{\mathsf{T}})$ using $(*)$, which in turn yields $(\mathsf{Op}^+_{(1,y)(0)} \circ \mathsf{Op}^+_{(1,y)(\alpha)})\mathsf{p}_1(\breve{\mathsf{T}})$, which is $\mathsf{Op}^+_{(1,y)(\alpha+1)}\mathsf{p}_1(\breve{\mathsf{T}})$.

Now, let us anticipate that for each n, "$(\mathsf{p}_3^n\mathsf{p}_1)$ proves $(\mathsf{It}_3^n, \mathsf{It}, \mathsf{it})$". Hence, it is plausible that "$\forall n(\mathsf{p}_2\mathsf{p}_3^n\mathsf{p}_2)^n\mathsf{p}_1$ prove $\bigcap_n(\mathsf{It} \circ \mathsf{It}_3^n(\mathsf{It}), \mathsf{It}, \mathsf{it})$". However, $\bigcap_n(\mathsf{It} \circ \mathsf{It}_3^n(\mathsf{It}), \mathsf{It}, \mathsf{it})$ is $(\mathsf{It}_3^\omega, \mathsf{It}, \mathsf{it})$. This indicates that $(\forall n(\mathsf{p}_2\mathsf{p}_3^n\mathsf{p}_2))^\alpha\mathsf{p}_1$ could prove $(\mathsf{It}_3^\omega(\mathsf{It}))^\alpha(\mathsf{it})$, which we take as evidence that $\mathsf{p}_3\mathsf{p}_1$ corresponds to $(\mathsf{It}_3^{\omega+1}, \mathsf{It}, \mathsf{it})$. That is, the operations with name $(1,(1,(\omega,q_0)))$ corresponds to the functional with name $(1,(1,(\omega{+}1,q_0)))$.

To conclude, we hint at some further extension. We defined the partial function $\circ$ so that $\mathsf{Op}_x \circ \mathsf{Op}_y$ iff $\mathsf{Op}_{x \circ y}$, and $H_x \circ H_y = H_{x \circ y}$. Now we further consider partial functions $\circ_m$ for each $m \in \mathbb{N}$ so that (given $x \circ_m y$ is defined) $\mathsf{Op}_x^{+m} \circ \mathsf{Op}_y$ iff $\mathsf{Op}_{x \circ_m y}$. We will see that if $\deg(x) = 1$, then Op_x iff $\mathsf{Op}_y \circ \mathsf{Op}_z$, and if $\deg(x) = m{+}2$, then Op_x iff $\mathsf{Op}_y^{+m} \circ \mathsf{Op}_z$, where in the first case, y is a simple name of degree one (of the form q_1 or (γ, v) or $(1{+}\beta, y_1^-)$, and in the second case, y is a simple name of degree two (of the form $(1, y')$, where $\deg(y') = 1$).

With functionals, application and composition are different. Therefore, we cannot directly form a new functionals out of e.g. H_x^{+2} and H_y. When working with functionals, it proves useful to write a name as $x = L(x_0 \circ_1 (1, y_0) \ldots \circ_1 (1, y_m)) \in Q^H$, where $L(y_1 \circ_1 \ldots \circ_1 y_m)$ indicates that we associate to the left. Then, it turns out that

$$H_x^{+n} = (H_{x_0}^{+(n+m)}, H_{(1,y_0)}^{+(n+m-1)}, \ldots, H_{(1,y_m)}^{+(n+0)}).$$

This form also works nicely with operations, where we have that for $x = L(x_0 \circ_1 (1, y_0) \circ_1 (1, y_1) \circ_1 \ldots \circ_1 (1, y_m)) \in Q$,

$$\mathsf{Op}_x^{+n} \Leftrightarrow \mathsf{Op}_{x_0}^{+(n+m)} \circ \mathsf{Op}_{(1,y_0)}^{+(n+m-1)} \circ \ldots \circ \mathsf{Op}_{(1,y_m)}^{+n}.$$

This will help to lift the notion "Op_x^{+} proves $H_{x^*}^{+}$" to "Op_x^{+n} proves $H_{x^*}^{+n}$" as mentioned at the beginning of this chapter.

III.1 Names

Now we introduce names in a more general form, still relying on Definition II.1.9 and Definition II.1.1 from Section II.1: names over an ordered set $(X, <)$ are finite sequences $\langle(\alpha_1, x_1), \ldots, (\alpha_k, x_k)\rangle$ of pairs $(\alpha_i, x_i) \in (\Omega\backslash\{0\})\times X$, so that $x_1 < \ldots < x_k$.

Definition III.1.1. *Let $(X, <)$ be an ordered set, and $(\Omega \times X, \lessdot)$ the ordering with $\langle\rangle \lessdot (\alpha, x) \lessdot (\beta, y)$ iff $x < y \vee (x = y \wedge \alpha < \beta)$. Then,*

$$\mathsf{name}(X) := \{\langle(1+\alpha_1, x_1), \ldots, (1+\alpha_k, x_k)\rangle \in (\Omega \times X)^{<\omega} : x_1 < \ldots < x_k\},$$

and $\mathsf{name}(X, <)$ is the ordered set $(\mathsf{name}(X), \lessdot_{lex})$.

Note that the empty sequence $\langle\rangle$ is a name above any set. The following is readily observed.

Lemma III.1.2. *If $(X, <)$ is a well-ordering, then also $(\mathsf{name}(X), \lessdot_{lex})$.*

If x and y are names over X, then $x * y$ is the concatenation of these finite sequences over $(\Omega \times X)$. In general, this is only a finite sequence, but not a name.

As in the case $N_0 = 2$, we will assign functionals to names so that $H_{\langle(\alpha,x)\rangle} \circ H_{\langle(\beta,x)\rangle} = H_{\langle(\beta+\alpha,x)\rangle}$, and accordingly for operations. This motivates to extend the partial function $* : \mathsf{name}(X)^2 \to \mathsf{name}(X)$ to a partial function $\circ : \mathsf{name}(X)^2 \to \mathsf{name}(X)$, so that $\langle(\alpha, x)\rangle \circ \langle(\beta, x)\rangle = \langle(\beta+\alpha, x)\rangle$.

Definition III.1.3. *Let x and y be names over X. If $\langle\rangle \in \{x, y\}$, then $x \circ y := x * y$, and if $x = \langle x_1, \ldots, x_k\rangle$ with $x_k = (\alpha, v)$ and $y = \langle y_1, \ldots, y_l\rangle$ with $y_1 = (\beta, w)$, then*

$$x \circ y :\simeq \begin{cases} x * y & : \text{ if } x * y \in \mathsf{name}(X), \\ \langle x_1, \ldots, x_{k-1}, (\beta+\alpha, v), y_2, \ldots, y_l\rangle & : \text{ if } v = w, \\ \uparrow & : \text{ else.} \end{cases}$$

Observe that if x and y are names and $x \circ y$ is defined, then $x \circ y$ is a name. Also the following is readily checked.

Lemma III.1.4.

(i) $\circ$ *is associative, that is,* $(x \circ y) \circ z \simeq x \circ (y \circ z)$.

(ii) If $y \neq \langle\rangle$*, then* $(x * y) \circ z \simeq x * (y \circ z)$ *and* $(x \circ y) * z \simeq x \circ (y * z)$.

Note that the assumption $y \neq \langle\rangle$ in (ii) is required, as for $y = \langle\rangle$, the claim reads $x \circ z \simeq x * z$, which does not hold in general.

The ordered set $(Q^H, <^H)$ and $(Q, <)$, which will be used to name functionals and operations, respectively, are such that $(Q^H, <^H) = \mathsf{name}(Q^H, <^H)$, and slightly more general, $(Q, <) = \mathsf{name}(Q', <')$, where $Q \subseteq Q'$, and $<=<'\restriction Q$ (you may want to peek at Definitions III.2.1 and III.3.1). With such a situation in mind, we define partial operations $\circ_n : \mathsf{name}(X)^2 \to \mathsf{name}(X)$ as follows.

Definition III.1.5. $\circ_0 := \circ$*, and for each* $n \in \mathbb{N}$ *and all* $x, y \in \mathsf{name}(X)$*,*

$$x \circ_{n+1} y :\simeq \begin{cases} \langle(1, x \circ_n z)\rangle \circ w & : \textit{if } y = \langle(1, z)\rangle * w \textit{ and } z \in \mathsf{name}(X), \end{cases}$$

For instance, if $x \neq \langle\rangle$ and $y := \langle(1, \langle\rangle), (1, x)\rangle$, then $y = \langle(1, \langle\rangle)\rangle * \langle(1, x)\rangle$. By definition, $x \circ_1 y = \langle(1, x \circ \langle\rangle)\rangle \circ \langle(1, x)\rangle$ which equals $\langle(2, x)\rangle$. This illustrates the reason for writing $\langle(1, x \circ_n z)\rangle \circ w$ (as opposed to $\langle(1, x \circ_n z)\rangle * w$) in the first clause of the above definition.

By induction on n, it is readily seen that if x and y are names and $x \circ_n y$ is defined, then $x \circ_n y$ is a name. Further, $x \circ_{n+1} (\langle(1, z)\rangle * w)$ is defined iff $v := \langle(1, x \circ_n z)\rangle$ is defined and if $v \circ w$ is defined.

Lemma III.1.6. *Assume that* $(X, <) = \mathsf{name}(X', <')$ *with* $X \subseteq X'$ *and* $<=<'\restriction X$*. Then, we have for all* $x', x, y \in X$ *and all* $m \in \mathbb{N}$*,*

if $x \circ_m y$ *is defined and* $x' < x$*, then* $x' \circ_m y < x \circ_m y$ *(so* $x' \circ_m y$ *is defined, too).*

Proof By induction on m. If $x \circ_0 y$ is defined, either $x * y \in X$, and then also $x' * y \in X$ and $x' * y < x * y$, or $(x)_{\mathsf{lh}(x)-1}$ is of the form (α, z) and $(y)_0$ is of the form (β, z), and then $x' < x$ implies $(x')_{\mathsf{lh}(x')-1} \leq (x)_{\mathsf{lh}(x')-1}$, hence $x' \circ y$ is defined and $x' \circ y < x \circ y$. And if $x \circ_{m+1} y$ is defined, then y is of the form $\langle(1, y')\rangle * w$ for some $y' \in X$, and $x \circ_{m+1} y = \langle(1, x \circ_m y')\rangle \circ w$. By I.H., $x' \circ_m y'$ is defined and $x' \circ_m y' < x \circ_m y'$. Since $<=<'\restriction X$, also $x' \circ_m y' <' x \circ_m y'$, hence $\langle(1, x' \circ_m y')\rangle < \langle(1, x \circ_m y')\rangle$, and by the case $m = 0$, $\langle(1, x' \circ_m y')\rangle \circ w < \langle(1, x \circ_m y')\rangle \circ w$. The claim follows. $\square$

Lemma III.1.7. *Assume that* $(X,<) = \mathsf{name}(X',<')$ *with* $X \subseteq X'$ *and* $<=<'\restriction X$, *and that* $x,y,w \in X$ *with* $x \neq \langle\rangle$. *Then,*

$$\langle(1, x \circ_m y)\rangle \circ w \text{ is defined iff } x \circ_{m+1} (\langle(1,y)\rangle * w) \in X.$$

Proof If $\langle(1, x \circ_m y)\rangle \circ w$ is defined, then $(w)_0$ is of the form (α, z) with $x \circ_m y \leq' z$, thus also $x \circ_m y \leq z$. Since $\langle\rangle < x$, we have $\langle\rangle \circ_m y = y < x \circ_m y$ by the above lemma. Since $<=<'\restriction X$, also $y <' x \circ_m y \leq' z$. Thus $\langle(1,y)\rangle * w \in X$. It follows that $x \circ_{m+1} (\langle(1,y)\rangle * w)$ is defined and an element of X. Conversely, if $x \circ_{m+1} (\langle(1,y)\rangle * w) \in X$ and hence defined, then it equals $\langle(1, x \circ_m y)\rangle \circ w$ which is thus defined, too. □

Lemma III.1.8. *Assume that* $(X,<) = \mathsf{name}(X',<')$ *with* $X \subseteq X'$ *and* $<=<'\restriction X$. *For all* $x,y,z \in X$ *with* $x \neq \langle\rangle$, *and all* $m,n \in \mathbb{N}$,

(i) $(x \circ_m y) \circ z \simeq x \circ_m (y \circ z)$,

(ii) $(x \circ_m y) \circ_n z \simeq x \circ_{m+n} (y \circ_n z)$.

Proof (i) For $m = 0$ the claim is by Lemma III.1.4. If $(x \circ_{m+1} y) \circ z$ is defined, then there are names y', w so that $y = \langle(1,y')\rangle * w$. If $w \neq \langle\rangle$, then, using Lemma III.1.4 (i), we obtain $(x \circ_{m+1} y) \circ z = ((1, x \circ_m y') \circ w) \circ z = (1, x \circ_m y') \circ (w \circ z) =_{L.III.1.7} x \circ_{m+1} ((1,y') * (w \circ z)) = x \circ_{m+1} (y \circ z)$. Conversely, if $x \circ_{m+1} (y \circ z)$ is defined, then $y \circ z$ is of the form $\langle(1,y')\rangle * w'$, and it is readily checked that then w' is of the form $w \circ z$ for some w. Thus, $x \circ_{m+1} (y \circ z) = x \circ_{m+1} ((1,y') * (w \circ z))$, and the other direction follows as above. If $w = \langle\rangle$, the claim is shown similar but simpler. (ii) By induction on n. The case $n = 0$ is by (i). For the induction step, observe that if z is not of the form $\langle(1,z')\rangle * w$, then both sides are undefined. So we can assume that $z = \langle(1,z')\rangle * w$ and that $z' \in X$. Then, $(x \circ_m y) \circ_{n+1} z \simeq (x \circ_m y) \circ_{n+1} \langle(1,z')\rangle * w) \simeq \langle(1, (x \circ_m y) \circ_n z')\rangle \circ w \simeq_{IH} \langle(1, x \circ_{m+n} (y \circ_n z'))\rangle \circ w \simeq (x \circ_{m+n+1} \langle(1, y \circ_n z')\rangle) \circ w \simeq_{(i)} x \circ_{m+n+1} (\langle(1, y \circ_n z')\rangle \circ w) \simeq_{L.III.1.7} x \circ_{m+n+1} (y \circ_{n+1} (\langle(1,z')\rangle * w)) \simeq x \circ_{m+n+1} (y \circ_{n+1} z)$. □

As $\circ_{n+1}$ is not associative, it matters whether we associate to the left or to the right. To deal with both cases, we introduce the following abbreviations.

Definition III.1.9.

(i) $L() := R() := \langle\rangle$, $L(x) := R(x) := x$ *and*
$L(x_1 \circ_{m_1} x_2) := R(x_1 \circ_{m_1} x_2) := (x_1 \circ_{m_1} x_2)$.

(ii) $L(x_1 \circ_{m_1} \ldots \circ_{m_{k+2}} x_{k+3}) := (L(x_1 \circ_{m_1} \ldots \circ_{m_{k+1}} x_{k+2}) \circ_{m_{k+2}} x_{k+3})$.

(iii) $R(x_1 \circ_{m_1} \ldots \circ_{m_{k+2}} x_{k+3}) := (x_1 \circ_{m_1} R(x_2 \circ_{m_2} \ldots \circ_{m_{k+2}} x_{k+3}))$.

Further, if $m > n$, then e.g. $L(x_m \circ_1 \ldots \circ_1 x_n) := L()$.

The next lemma collects various properties that allow us to move from $L(\ldots)$ to $R(\ldots)$, and to compute $x \circ_n L(\ldots)$ and $R(\ldots) \circ_n x$.

Lemma III.1.10. *Assume that $(X, <) = \mathsf{name}(X', <')$ with $X \subseteq X'$ and $< = <' \restriction X$. For all $\vec{x} \in X$,*

(i) $R(x_0 \circ_{m_1} \ldots \circ_{m_k} x_k) \circ_n x_{k+1} \simeq R(x_0 \circ_{m_1+n} \ldots \circ_{m_k+n} x_k \circ_n x_{k+1})$,

(ii) $L(x_0 \circ_{m_1} \ldots \circ_{m_k} x_k) \simeq R(x_0 \circ_{M_1^k} x_1 \ldots \circ_{M_k^k} x_k)$,
where $M_k^k := m_k$ for $1 \leq i < k$, $M_i^k := m_k + \ldots + m_i$.

(iii) $x_0 \circ_{M_1^k} L(x_1 \circ_{m_2} \ldots \circ_{m_k} x_k) \simeq L(x_0 \circ_{m_1} x_1 \circ_{m_2} \ldots \circ_{m_k} x_k)$.

(iv) $L(x_0 \circ_{m_1} \ldots \circ_{m_k} x_k) \simeq R(x_0 \circ_{M_1^i} \ldots \circ_{M_i^i} x_i) \circ_{M_{i+1}^k} L(x_{i+1} \circ_{m_{i+2}} \ldots \circ_{m_k} x_k)$.

Proof (i) By induction on k. For $k = 0$ this is trivial and for $k = 1$ this is Lemma III.1.8 (ii). The induction step is shown as follows.

$$R(x_0 \circ_{m_1} \ldots \circ_{m_{k+1}} x_{k+1}) \circ_n x_{k+2} \simeq (x_0 \circ_{m_1} R(x_1 \circ_{m_2} \ldots \circ_{m_{k+1}} x_{k+1})) \circ_n x_{k+2} \simeq_{\text{L.III.1.8 (ii)}}$$
$$x_0 \circ_{m_1+n} (R(x_1 \circ_{m_2} \ldots \circ_{m_{k+1}} x_{k+1}) \circ_n x_{k+2}) \simeq_{IH}$$
$$x_0 \circ_{m_1+n} R(x_1 \circ_{m_2+n} \ldots \circ_{m_{k+1}+n} x_{k+1} \circ_n x_{k+2}) \simeq R(x_0 \circ_{m_1+n} \ldots \circ_{m_{k+1}+n} x_{k+1} \circ_n x_{k+2}).$$

(ii) By induction on $k \geq 1$. For $k = 1$, the claim is obvious, thus we show the induction step: $L(x_0 \circ_{m_1} \ldots \circ_{m_{k+1}} x_{k+1}) \simeq L(x_0 \circ_{m_1} \ldots \circ_{m_k} x_k) \circ_{m_{k+1}} x_{k+1} \simeq_{IH} R(x_0 \circ_{M_1^k} \ldots \circ_{M_k^k} x_k) \circ_{m_{k+1}} x_{k+1} \simeq_{(i)} R(x_0 \circ_{M_1^{k+1}} x_1 \ldots \circ_{M_k^{k+1}} x_k \circ_{M_{k+1}^{k+1}} x_{k+1})$.

(iii) $z := L(x_1 \circ_{m_2} \ldots \circ_{m_k} x_k) \simeq_{(ii)} R(x_1 \circ_{M_2^k} \ldots \circ_{M_k^k} x_k)$.

$$x_0 \circ_{M_1^k} z \simeq x_0 \circ_{M_1^k} R(x_1 \circ_{M_2^k} \ldots \circ_{M_k^k} x_k) \simeq$$
$$x_0 \circ_{M_2^k + m_1} (x_1 \circ_{M_2^k} R(x_2 \circ_{M_3^k} \ldots \circ_{M_k^k} x_k)) \simeq_{L.III.1.8(ii)}$$
$$(x_0 \circ_{m_1} x_1) \circ_{M_2^k} (R(x_2 \circ_{M_3^k} \ldots \circ_{M_k^k} x_k)) \simeq R((x_0 \circ_{m_1} x_1) \circ_{M_2^k} \ldots \circ_{M_k^k} x_k).$$

Using (ii) again yields the claim.

(iv) By induction on i. If $i = 0$, then we have $L(x_0 \circ_{m_1} \ldots \circ_1 x_k) \simeq_{(iii)} R(x_0) \circ_{M_1^k}$

$L(x_1 \circ_{m_2} \ldots \circ_{m_k} x_k)$. And if $0 < i = j+1$, we have for $z := R(x_0 \circ_{M_1^j} \ldots \circ_{M_j^j} x_j)$,

$$
\begin{aligned}
L(x_0 \circ_{m_1} \ldots \circ_{m_k} x_k) &\simeq_{IH} z \circ_{M_{j+1}^k} L(x_{j+1} \circ_{m_{j+2}} \ldots \circ_{m_k} x_k) \simeq_{(iii)} \\
&z \circ_{M_i^k} (x_{j+1} \circ_{M_{i+1}^k} L(x_{i+1} \circ_{m_{i+2}} \ldots \circ_{m_k} x_k)) \simeq_{L.III.1.8} \\
&(z \circ_{m_i} x_{j+1}) \circ_{M_{i+1}^k} L(x_{i+1} \circ_{m_{i+2}} \ldots \circ_{m_k} x_k) \simeq_{(i)} \\
&R(x_0 \circ_{M_1^{j+1}} \ldots \circ_{M_{j+1}^{j+1}} x_{j+1}) \circ_{M_{i+1}^k} L(x_{i+1} \circ_{m_{i+2}} \ldots \circ_{m_k} x_k).
\end{aligned}
$$

□

The following corollary singles out what is actually need in the sequel.

Corollary III.1.11. *Under the assumptions of the above lemma,*

(i) $L(x_1 \circ_1 x_2 \circ_1 \ldots \circ_1 x_{m-1} \circ_1 x_m) \circ z \simeq R(x_1 \circ_{m-1} x_2 \circ_{m-2} \ldots \circ_2 x_{m-1} \circ_1 x_m) \circ z$*,*

(ii) $x_0 \circ_{m+n} L(x_1 \circ_1 \ldots \circ_1 x_{m+1}) \circ z \simeq L((x_0 \circ_n x_1) \circ_1 \ldots \circ_1 x_{m+1}) \circ z$*.*

III.2 Names for functionals

First, we introduce an ordered set $(Q^H, <^H)$ of names. Essentially by its definition, we have that $\mathsf{name}(Q^H, <^H) = (Q^H, <^H)$. Then, we assign to each $x \in Q^H$ a functional H_x.

Definition III.2.1. $(Q_0^H, <_0^H) := (\{\langle\rangle\}, \emptyset)$*, and* $(Q_{n+1}^H, <_{n+1}^H) := \mathsf{name}(Q_n^H, <_n^H)$*. Then,* $(Q^H, <^H) := (\bigcup_n Q_n^H, \bigcup_n <_n^H)$*. Further, the least* n *so that* $x \in Q_n^H$ *is called the level* $\mathsf{lv}(x)$ *of* x*.*

That is, $Q_{n+1}^H = \{\langle(1+\alpha_1, x_1), \ldots, (1+\alpha_k, x_k)\rangle : x_1 <_n^H \ldots <_n^H x_k; x_1, \ldots, x_k \in Q_n^H\}$, and $<_{n+1}^H = \lessdot_{\mathrm{lex}} \restriction Q_{n+1}^H$, where by Definition II.1.1, $\lessdot$ is the ordering on $\Omega \times Q_n^H$ with $(\alpha, x) \lessdot (\beta, y)$ iff $x <_n^H y \vee (x = y \wedge \alpha < \beta)$.

By induction on n it is immediate that $Q_n^H \subseteq Q_{n+1}^H \subseteq Q^H$, thus $<_n^H = <^H \restriction Q_n^H$, and further, if $\mathsf{lv}(x) < \mathsf{lv}(y) = n$, then $x <_n^H y$. We just look at the induction step of the last claim: if $y \in Q_{n+1}^H \setminus Q_n^H$ and $x \neq \langle\rangle$, then $x = \langle(\alpha_1, x_1), \ldots, (\alpha_k, x_k)\rangle$ and $y = \langle(\beta_1, y_1), \ldots, (\beta_l, y_l)\rangle$ with $y_1 <_n^H \ldots <_n^H y_l$. Since $y \in Q_{n+1}^H \setminus Q_n^H$, the I.H. implies that $y_l \in Q_n^H$. Again by I.H., $x_k <_n^H y_l$, so $x <_{n+1}^H y$. And if $x = \langle\rangle$, then $x <_n^H y$ is directly by definition of $<_n^H$.

As names are used quite frequently in the sequel, in order to increase readability, we stick to the following abbreviations, and further, to Convention II.1.3.

Definition III.2.2. *We let* $q_0 := \langle\rangle$*,* $q_{n+1} := \langle(1, q_n)\rangle$ *and* $q_1^\alpha := \langle(\alpha, q_0)\rangle$*.*

Since q_0 is the only name with $\mathsf{lv}(q_0) = 0$, also the names of level 1 are very simple.

Lemma III.2.3. *If $x \in Q^H$ with $\mathsf{lv}(x) = 1$, then $x = \langle(\alpha, q_0)\rangle$ for some α. In particular, $\mathsf{lh}(x) = 1$.*

Now we assign to each name $x \in Q^H$ and each $n \in \mathbb{N}$ a functional of type-n+2, using our basic functionals It_{n+1} (cf. Definition I.3.10).

Definition III.2.4. *For each n, $H^{+n}_{q_0}$ is the identity on $\Omega^{(n)}$, and for $0 < \alpha, \beta < \Omega$ and $k > 1$,*

(i) $H^{+n}_{(\alpha,q_0)} := \mathsf{It}^{\alpha}_{n+1}$.

(ii) $H^{+n}_{(\beta,x)} := (H^{+(n+1)}_x(\mathsf{It}_{n+1}))^{\beta}$ $(x \neq q_0)$,

(iii) $H^{+n}_{\langle x_1,\ldots,x_k\rangle} := H^{+n}_{x_1} \circ \ldots \circ H^{+n}_{x_k}$.

Further, if $f \in \Omega^{(0)}$, then $f_x := H_x(f)$, and $H_x := H^{+0}_x$ and $H^+_x := H^{+1}_x$.

Next, we check that $H^{+n}_x \in \Omega^{(n+1)}$, and that $H^{+n}_x \subseteq \mathsf{It}_{n+1}$ (cf. Convention I.3.5), a simple but useful property. Then, we verify that $H^{+n}_{x \circ y} = H^{+n}_x \circ H^{+n}_y$. Finally, we reveal a point of writing a name in the form $x = L(y_0 \circ_1 \ldots \circ_1 y_{m+1})$.

Lemma III.2.5. *For each $n \in \mathbb{N}$ and each $x \in Q^H \setminus \{q_0\}$, $H^{+n}_x \in \Omega^{(n+1)}$ and $H^{+n}_x \subseteq \mathsf{It}_{n+1}$.*

Proof Recall that if $F \in \Omega^{(n+1)}$, then F is strictly inclusive, that is, $F(G) \subseteq G$ for each $G \in \Omega^{(n)}$, and further, $F^{1+\alpha} \in \Omega^{(n+1)}$ for each α (cf. Lemma I.3.12). Moreover, $F^{1+\alpha} \subseteq F$, as is readily seen by induction on α, and if $\beta \leq \alpha$, then $F^{\alpha} \subseteq F^{\beta}$.

The two claims are shown simultaneously by induction on the build up of Q^H. By Corollary I.3.13, $\mathsf{It}_{n+1} \in \Omega^{(n+1)}$. Hence, $H^{+n}_{(1+\alpha,q_0)} = (\mathsf{It}_{n+1})^{1+\alpha} \in \Omega^{(n+1)}$ and $(\mathsf{It}_{n+1})^{1+\alpha} \subseteq \mathsf{It}_{n+1}$. If both claims hold for $x \in Q^H \setminus \{q_0\}$, then $H^{+(n+1)}_x \in \Omega^{(n+2)}$ and $H^{+(n+1)}_x \subseteq \mathsf{It}_{n+2}$, hence also $H^{+(n+1)}_x(\mathsf{It}_{n+1}) \in \Omega^{(n+1)}$ and

$$H^{+n}_{(1+\alpha,x)} = (H^{+(n+1)}_x(\mathsf{It}_{n+1}))^{1+\alpha} \subseteq H^{+(n+1)}_x(\mathsf{It}_{n+1}) \subseteq \mathsf{It}_{n+2}(\mathsf{It}_{n+1}) \subseteq \mathsf{It}_{n+1}.$$

And if both claims hold for x_k, and $x = \langle x_1, \ldots, x_k\rangle$ and $k > 1$, then $H^{+n}_{\langle x_1,\ldots,x_k\rangle} = H^{+n}_{x_1} \circ \ldots \circ H^{+n}_{x_k} \subseteq H^{+n}_{x_k} \subseteq \mathsf{It}_{n+1}$, and $H^{+n}_{\langle x_1,\ldots,x_k\rangle} \in \Omega^{(n+1)}$ as $\Omega^{(n+1)}$ is closed under composition. $\square$

Lemma III.2.6. *For each $x \in Q^H \setminus \{q_0\}$, $\mathsf{It}_{n+2}(H^{+n}_x) \subseteq \mathsf{It}_{n+1} \circ H^{+n}_x$.*

Proof

$$(\mathsf{It}_{n+2}(H_x^{+n}),\ldots,f,\alpha) = \mathsf{It}_{n+2}[H_x^{+n},\ldots,f,\alpha] \in ((H_x^{+n})^{2+\alpha},\ldots,f) \subseteq$$
$$((H_x^{+n})^2,\ldots,f) \subseteq_{L.III.2.5} ((\mathsf{It}_{n+1}\circ H_x),\ldots,f) = (\mathsf{It}_{n+1}\circ H_x)[\ldots,f].$$

□

Lemma III.2.7. *For all $x_0,\ldots,x_{m+2} \in Q^H \setminus \{q_0\}$,*

$$\mathsf{It}_{m+2}[H_{x_m}^{+m},\ldots,H_{x_0}^{+0}] \subseteq \mathsf{it}\circ(H_{x_m}^{+m},\ldots,H_{x_0}^{+0}).$$

Proof By induction on m. For $m=0$ this is by the above lemma, and

$$\mathsf{It}_{m+3}[H_{x_{m+1}}^{+(m+1)},\ldots,H_{x_0}^{+0}] = (\mathsf{It}_{m+3}(H_{x_{m+1}}^{+(m+1)}),\ldots,H_{x_0}^{+0}) \subseteq_{L.III.2.6}$$
$$((\mathsf{It}_{m+2}\circ H_{x_{m+1}}^{+(m+1)}),\ldots,H_{x_0}^{+0}) = \mathsf{It}_{m+2}[H_{x_{m+1}}^{+(m+1)}(H_{x_m}^{+m}),\ldots,H_{x_0}^{+0}] \subseteq_{IH}$$
$$\mathsf{it}\circ(H_{x_{m+1}}^{+(m+1)}(H_{x_m}^{+m}),\ldots,H_{x_0}^{+0}) = \mathsf{it}\circ[H_{x_{m+1}}^{+(m+1)},\ldots,H_{x_0}^{+0}].$$

□

Lemma III.2.8. *If $x\circ y \in Q^H$, then $H_{x\circ y}^{+n} = H_x^{+n}\circ H_y^{+n}$.*

Proof Immediate by Definition III.2.4 and the observation that for any functional $F \in \Omega^{(n+1)}$, $F^\alpha \circ F^\beta = F^{\beta+\alpha}$ (cf. Lemma I.3.14). □

Lemma III.2.9. *Let $x = L(y_0 \circ_1 \ldots \circ_1 y_m) \in Q^H$ with $\mathsf{lh}(y_i) = 1$ $(0 \le i \le m)$. Then,*

$$H_x^{+n} = (H_{y_0}^{+(n+m)}, H_{y_1}^{+(n+m-1)},\ldots,H_{y_m}^{+(n+0)}).$$

Proof By induction on m. For $m=0$ there is nothing to show. And if $x = L(y_0\circ_1\ldots\circ_1 y_{m+1}) \in Q^H$, then $x = L(y_0\circ_1\ldots\circ_1 y_m)\circ_1 y_{m+1}$. Since this is defined, y_{m+1} is of the form $(1,z)$, and therefore $x = (1, L(y_0\circ_1\ldots\circ_1 y_m)\circ z)$. Hence,

$$\begin{aligned}
H_x^{+n} &= H^{+n}_{(1,L(y_0\circ_1\ldots\circ_1 y_m)\circ z)} = H^{+(n+1)}_{L(y_0\circ_1\ldots\circ_1 y_m)\circ z}(\mathsf{It}_{n+1}) \\
&= (H^{+(n+1)}_{L(y_0\circ_1\ldots\circ_1 y_m)}\circ H_z^{+(n+1)})(\mathsf{It}_{n+1}) = H^{+(n+1)}_{L(y_0\circ_1\ldots\circ_1 y_m)}(H_z^{+(n+1)}(\mathsf{It}_{n+1})) \\
&= H^{+(n+1)}_{L(y_0\circ_1\ldots\circ_1 y_m)}(H^{+n}_{(1,z)}) = H^{+(n+1)}_{L(y_0\circ_1\ldots\circ_1 y_m)}(H^{+n}_{y_{m+1}}) \\
&=_{IH} (H_{y_0}^{+(n+m+1)}, H_{y_1}^{+(n+m)},\ldots,H_{y_m}^{+(n+1)})(H^{+n}_{y_{m+1}}) \\
&= (H_{y_0}^{+(n+m+1)}, H_{y_1}^{+(n+m)},\ldots,H_{y_m}^{+(n+1)}, H^{+n}_{y_{m+1}}).
\end{aligned}$$

□

III.3 Names for operations

In this section, we present an ordered set $(Q,<)$ of names for operations, and assign to each $q_0 \neq x \in Q$ an operation Op_x. However, as the situation with operations is more complex than with functionals, we no longer have that $\mathsf{name}(Q,<) = (Q,<)$, but only that $\mathsf{name}(Q \cup P,<) = (Q,<)$, where $P \subseteq Q^-$ (cf. Definition II.1.9) is a set of prenames. Prenames are not names, but used to form names.

Definition III.3.1. *$(Q_0,<_0) := (\{\langle\rangle\},\emptyset)$, and $(Q_{n+1},<_{n+1}) := \mathsf{name}(Q_n \cup P_n,<_n)$, where $P_0 = \emptyset$, and $P_{n+1} \subseteq Q^-_{n+1}$ so that*

$$P_{n+1} = \{z^- : (z)_0 = (\gamma, y), y \in Q_n\} \cup \{z^- : (z)_0 = (1{+}\alpha, v) : v \in P_n\}.$$

Then, we set $(Q,<) := (\bigcup_n Q_n, \bigcup_n <_n)$ and $(P,<) := (\bigcup_n P_n, \bigcup_n <_n)$. Further, $Q^*_n := Q_n \setminus \{q_0\}$, $Q^* := Q \setminus \{q_0\}$, and the least n so that $v \in Q_n \cup P_n$ is called the level $\mathsf{lv}(v)$ of v.

Note that when switching from $(Q_n,<_n)$ to $(Q_n \cup P_n,<_n)$, the ordering $<_n$ is extended according to Definition II.1.9, and that $<_{n+1}= \lessdot_{\mathrm{lex}}\restriction(Q_{n+1} \cup P_{n+1})$, where $\lessdot$ is the ordering on $\Omega \times (Q_n \cup P_n)$ with $(\alpha, v) \lessdot (\beta, w)$ iff $v <_n w \vee (v = w \wedge \alpha < \beta)$.

This time, $Q_{n+1} = \{\langle(1{+}\alpha_1, v_1), \ldots, (1{+}\alpha_k, v_k)\rangle : v_1 <_n \ldots <_n v_k \in Q_n \cup P_n\}$, and $x^- \in P_{n+1}$ iff $x \in Q_{n+1}$ and $(x)_0$ is either of the form (γ, z) with $z \in Q_n$, or of the form $(1{+}\alpha, v)$ with $v \in P_n \subseteq Q^-_n$.

As with Q^H, we have that $Q_n \subseteq Q_{n+1} \subseteq Q$, $P_n \subseteq P_{n+1} \subseteq P$, $<_n=<\restriction Q_n$, and $(Q_n \cup P_n,<_n)$ is according to Definition II.1.9. If $\mathsf{lv}(v) < \mathsf{lv}(w) = n$, then $v <_n w$. Also note that if $x \in Q$ with $\mathsf{lv}(x) \leq 1$, then $x \in Q^H$. Further, if $z^- \in P$, then z is the $<$-least element above z^-. Moreover, if $<_n\restriction(Q_n \cup P_n)$ is a well-ordering, then $\lessdot_{\mathrm{lex}}$ is a well-ordering on $\Omega \times (Q_n \cup P_n)$, and so $<_{n+1}\restriction(Q_{n+1} \cup P_{n+1})$ is a well-ordering, too. Therefore, $(Q,<)$ and $(Q \cup P,<)$ are well-orderings.

We extend Convention II.1.3 as follows.

Convention III.3.2. *We let x, y, z range over Q, and v, w over $Q \cup P$. If we write $v^- \in P$, then it is understood that $v \in Q$ and $v^- \in P$. Further, we write $(\alpha, v)^-$ for $\langle(\alpha, v)\rangle^-$, and moreover, if $z^- \in P$ and $x \in Q$, then $z^- * x := (z * x)^-$.*

Definition III.3.3. *We let $q_0 := \langle\rangle$ and $q_{n+1} := (1, q_n)$.*

Provisional definitions of the operations $(\mathsf{Op}_x : x \in Q^*)$

As in the case $N_0 = 2$, we give first a provisional definition of the operations $(\mathsf{Op}_x : x \in Q^*)$. Again, this definition is semantical: given $x \in Q^*$, Op_x is an operation, and it is assumed that we can represent this operation by an $\mathsf{L}_2(\mathsf{P})$-sentence, by using some way to code x as a natural number. Later on (after introducing approximations and normal forms for names in Q), it is superseded by Definition III.6.1, the proper definition of the operations $(\mathsf{Op}_x^{+n} : x \in Q^*)$, which provides for each $n \in \mathbb{N}$ an $\mathsf{L}_2(\mathsf{P})$-formulas $\varphi^{+n}(u)$ so that $\mathsf{T} \mapsto \varphi^{+n}(x)\{\mathsf{T}{\restriction}U\}$ is the operation Op_x^{+n}. There, it is assumed that we have a primitive recursive relation which codes Q^*, which is also denoted by Q^*.

From a technical point of view, none of the remainder of this section is henceforth required. It purpose is solely to convey some intuition of how the operations Op_x look like. Therefore, some proofs are a bit sketchy.

The next definition is by recursion on the build-up of $Q \cup P$, and is structured as follows. The first group of clauses says what operation is assigned to a name in Q, the second says what operation is assigned to a prename in P. But the two groups are interdependent.

Definition III.3.4. *For each n, all $\langle x_1, \ldots, x_k\rangle, x \in Q^*$ $(k > 1)$, each $y^- \in P$, each $v \in Q^* \cup P$ and each $\alpha > 0$, we have*

(i) $\mathsf{Op}_{q_1}^{+n} := \mathsf{p}_{n+1}$,

(ii) $\mathsf{Op}_{(m+1,x)}^{+n} := (\mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1})^{m+1}$ *and* $\mathsf{Op}_{(\gamma+m+1,x)}^{+n} := (\mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1})^{\gamma+m}$,

(iii) $\mathsf{Op}_{(\gamma,x)}^{+n} := (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^{\gamma}$,

(iv) $\mathsf{Op}_{(1+\alpha,y^-)}^{+n} := (\mathsf{Op}_{y^-}^{+(n,n+1)}\mathsf{p}_{n+1})^{1+\alpha}$

(v) $\mathsf{Op}_{\langle x_1,\ldots,x_k\rangle}^{+n} := \mathsf{Op}_{x_1}^{+n} \circ \ldots \circ \mathsf{Op}_{x_k}^{+n}$,

and for all m, n with $0 \le m < n$, we have

(iii)' $\mathsf{Op}_{(\gamma,v)^-}^{+(m,n)} := (\forall\alpha < \gamma)\mathsf{p}_{m+1}(\mathsf{Op}_{(1,v)}^{+n})^{1+\alpha}$,

(vi)' $\mathsf{Op}_{(\alpha+1,y^-)^-}^{+(m,n)} := \mathsf{Op}_{y^-}^{+(m,n+1)}\mathsf{p}_{n+1} \circ \mathsf{Op}_{(\alpha,y^-)}^{+n}$,

(v)' $\mathsf{Op}_{\langle y_1,\ldots,y_k\rangle^-}^{+(m,n)} := \mathsf{Op}_{y_1^-}^{+(m,n)} \circ \mathsf{Op}_{y_2}^{+n} \circ \ldots \circ \mathsf{Op}_{y_k}^{+n}$.

Further, $\mathsf{Op}_x := \mathsf{Op}_x^{+0}$, *and* $\mathsf{Op}_x^{+} := \mathsf{Op}_x^{+1}$.

(i)-(v) and (iii)' generalize Definition II.1.12; the clauses (vi)' and (v)' have no correspondence, as $x^- \in P_1$ iff $x = (\gamma, q_0)$. The role of the extra parameter m becomes visible in clause (iii)'. Also note that $\mathsf{Op}_{(\alpha+1,y^-)^-}^{+(m,n)} \Leftrightarrow \mathsf{Op}_{(1,y^-)^-}^{+(m,n)} \circ \mathsf{Op}_{(\alpha,y^-)}^{+n}$, that is, would we extend $\circ$ to $Q \cup P$, then $(1, v)^- \circ (\alpha, v) := (\alpha{+}1, v)^-$. Similar with clause (v)', which is the reason for letting $z^- * x := (z * x)^-$ (cf. Convention III.3.2).

Lemma III.3.5. *Let $y^- \in P$ and $x \in Q^*$ with $\deg(x) = m+1$. Then,*

(i) for all $0 \le k < n$, $\mathsf{Op}_{y^-}^{+(k,n)}(\breve{\mathsf{T}})$ is Π^1_{k+2},

(ii) for all n, $\mathsf{Op}_x^{+n}(\breve{\mathsf{T}})$ is Π^1_{m+n+2}.

Proof (i) By induction on the definition of $\mathsf{Op}_{y^-}^{+(k,n)}$. Note that $\mathsf{Op}_{(\gamma,v)^-}^{+(k,n)}(\breve{\mathsf{T}})$ is Π^1_{k+2} by definition, since $\mathsf{Op}_{x^-}^{+(k,n)}(\breve{\mathsf{T}})$ is of the form $\forall \alpha A(\alpha)$, where A is Π^1_{k+2}. In the other cases, the I.H. applies directly. (ii) By induction on the definition of Op_x^{+n} using (i). $\square$

To gain some intuition for the operations Op_x^{+n}, we state some relevant properties. For the time being, we just add some proof-sketches. Rigorous proofs are provided once we have introduced the proper definition of these operations.

Lemma III.3.6.

(i) If $0 < k \le m < n$, then $\mathsf{p}_n\mathsf{p}_k \Leftrightarrow \mathsf{p}_n\mathsf{p}_m\mathsf{p}_k$.

(ii) If $x \in Q^*$, then $\mathsf{Op}_x^{+n} \Rightarrow \mathsf{p}_{n+1}$.

Proof (i) If $0 < k \le m < n$, then $\mathsf{p}_n\mathsf{p}_k(\breve{\mathsf{T}})$ implies $\mathsf{p}_m(\mathsf{p}_k(\breve{\mathsf{T}}))$ which is Π^1_{m+1}, thus Lemma I.2.14 yields $\mathsf{p}_n\mathsf{p}_m\mathsf{p}_k(\breve{\mathsf{T}})$. Conversely, $\mathsf{p}_m\mathsf{p}_k \Rightarrow \mathsf{p}_k$ is by Lemma I.2.12, and since p_n is an operation, $\mathsf{p}_n\mathsf{p}_m\mathsf{p}_k \Rightarrow \mathsf{p}_n\mathsf{p}_k$ follows. (ii) By induction on the definition of Op_x^{+n}. $\square$

For instance, $\mathsf{p}_3\mathsf{p}_2\mathsf{p}_1 \Leftrightarrow \mathsf{p}_3\mathsf{p}_1$.

The following is relevant in particular for $n > 0$. It generalizes Lemma I.2.14. For a proof, we refer to Lemma III.6.6.

Lemma III.3.7. *For each open Π^1_{2+n}-sentence C, $\mathsf{T}^\epsilon \vdash \mathsf{Op}^{+n}(\breve{\mathsf{T}}) \wedge C \to \mathsf{Op}_x^{+n}(C)$.*

A typical application of this lemma is the proof of the right-to-left direction of the following lemma, which is the critical case in showing that $\mathsf{Op}_x \circ \mathsf{Op}_x \Leftrightarrow \mathsf{Op}_{x \circ y}$ (see Lemma III.3.9).

Lemma III.3.8. *If $(1, v) \in Q^*$, then $\mathsf{Op}_{(1,v)} \circ \mathsf{Op}_{(\gamma,v)} \Leftrightarrow \mathsf{Op}_{(\gamma+1,v)}$.*

Proof By definition, we have $\mathsf{Op}_{(1,v)} \Leftrightarrow \mathsf{Op}_v^+\mathsf{p}_1$, $\mathsf{Op}_{(\gamma,v)} \Leftrightarrow (\forall \xi < \gamma)\mathsf{p}_1(\mathsf{Op}_v^+\mathsf{p}_1)^{1+\xi}$, and $\mathsf{Op}_{(\gamma+1,v)} \leftrightarrow (\mathsf{Op}_v^+\mathsf{p}_1)^\gamma$. As $\mathsf{Op}_v^+\mathsf{p}_1$ is an operation, $\mathsf{Op}_v^+\mathsf{p}_1 \circ (\forall \xi < \gamma)\mathsf{p}_1(\mathsf{Op}_v^+\mathsf{p}_1)^{1+\xi}$ implies $(\forall \xi < \gamma)\mathsf{Op}_v^+\mathsf{p}_1\mathsf{p}_1(\mathsf{Op}_v^+\mathsf{p}_1)^{1+\xi}$, which in turn yields $(\forall \xi < \gamma)(\mathsf{Op}_v^+\mathsf{p}_1)^{1+\xi+1}$, so $(\mathsf{Op}_v^+\mathsf{p}_1)^\gamma$ follows.

For the converse direction, note that $(\forall \xi < \gamma)(\mathsf{Op}_v^+\mathsf{p}_1)^{\xi+1} \Rightarrow (\forall \xi < \gamma)\mathsf{p}_1(\mathsf{Op}_v^+\mathsf{p}_1)^{\xi+1}$, so $\mathsf{Op}_{(\gamma+1,v)}(\breve{\mathsf{T}})$ implies $\mathsf{Op}_{(\gamma,v)}(\breve{\mathsf{T}})$. And clearly, $(\forall \xi < \gamma)(\mathsf{Op}_v^+\mathsf{p}_1)^{\xi+1} \Rightarrow (\mathsf{Op}_v^+\mathsf{p}_1)$. By the above lemma, $\mathsf{Op}_{(1,v)}(\breve{\mathsf{T}}) \wedge \mathsf{Op}_{(\gamma,v)}(\breve{\mathsf{T}}) \to (\mathsf{Op}_{(1,v)} \circ \mathsf{Op}_{(\gamma,v)})(\breve{\mathsf{T}})$, as $\mathsf{Op}_{(\gamma,v)}(\breve{\mathsf{T}})$ is Π^1_2 and $\mathsf{Op}_{(1,v)}(\breve{\mathsf{T}})$ iff $\mathsf{Op}^+_v(\mathsf{p}_1(\breve{\mathsf{T}}))$. The claim follows. $\square$

Recall that $x \circ (1,v)$ is only defined if $v \in Q$. Then, $x \circ (1,v) = (1,(x \circ v))$.

Lemma III.3.9. *Let $x, y, x \circ y \in Q^*$. Then,*

(i) $\mathsf{Op}^{+n}_{x \circ y} \Leftrightarrow \mathsf{Op}^{+n}_x \circ \mathsf{Op}^{+n}_y$, *and*

(ii) $\mathsf{Op}^{+n}_{x \circ_m y} \Leftrightarrow \mathsf{Op}^{+(m+n)}_x \circ \mathsf{Op}^{+n}_y$.

Proof (i) is readily shown by induction on the definition of Op_x using Lemma III.3.8. (ii) By induction on m. The case $m = 0$ is by (i). For the induction step, note that if $x \circ_{m+1} y \in Q^*$, then $y = (1,z)$ or $y = (1,z) * z'$ for some $z, z' \in Q$. It suffices to show the claim for $y = (1,z)$, the general case is then by (i). If $z \neq q_0$, we have $\mathsf{Op}^{+n}_{x \circ_{m+1} y} = \mathsf{Op}^{+n}_{(1, x \circ_m z)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x \circ_m z}\mathsf{p}_{n+1} \Leftrightarrow_{IH} (\mathsf{Op}^{+(m+n+1)}_x \circ \mathsf{Op}^{+(n+1)}_z)\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}^{+(m+n+1)}_x \circ \mathsf{Op}^{+n}_{(1,z)}$. If $z = q_0$, then $m = 1$, and the claim follows as $\mathsf{Op}^{+n}_{x \circ_1 y} = \mathsf{Op}^{+n}_{(1,x)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_x \circ \mathsf{p}_{n+1} \Leftrightarrow_{m=1} \mathsf{Op}^{+(m+n)}_x \circ \mathsf{Op}^{+n}_y$. $\square$

The following corresponds to Lemma III.2.9. Note that below, no assumption $\mathsf{lh}(y_i)$ is required. This is owed to the fact that in contrast to functionals, with operation there is no difference between composition and application, i.e. $(\mathsf{Op}^+_x \circ \mathsf{Op}_y) \circ \mathsf{Op}_z \Leftrightarrow \mathsf{Op}^+_x \circ (\mathsf{Op}_y \circ \mathsf{Op}_z)$, whereas $H^+_x(H_y) \circ H_z$ may be different from $H^+_x(H_y \circ H_z)$.

Lemma III.3.10. *Let $x = R(y_0 \circ_m y_1 \circ_{m-1} \ldots \circ_1 y_m) \in Q^*$. Then,*

$$\mathsf{Op}^{+n}_x \Leftrightarrow \mathsf{Op}^{+(n+m)}_{y_0} \circ \mathsf{Op}^{+(n+m-1)}_{y_1} \circ \ldots \circ \mathsf{Op}^{+(n+0)}_{y_m}.$$

Proof By induction on m. For $m = 0$ there is nothing to show. And if $x = R(y_0 \circ_{m+1} \ldots \circ_1 y_{m+1})$, then $x = y_0 \circ_{m+1} R(y_1 \circ_m \ldots \circ_1 y_{m+1})$, hence

$$\begin{aligned} \mathsf{Op}^{+n}_x \quad &\Leftrightarrow \quad \mathsf{Op}^{+n}_{y_0 \circ_{m+1} R(y_1 \circ_m \ldots \circ_1 y_{m+1})} \Leftrightarrow_{L.III.3.9} \mathsf{Op}^{+n+(m+1)}_{y_0} \circ \mathsf{Op}^{+n}_{R(y_1 \circ_m \ldots \circ_1 y_{m+1})} \\ &\Leftrightarrow_{IH} \quad \mathsf{Op}^{+n+(m+1)}_{y_0} \circ \mathsf{Op}^{+(n+m)}_{y_1} \circ \mathsf{Op}^{+(n+m-1)}_{y_2} \circ \ldots \circ \mathsf{Op}^{+n}_{y_{m+1}}. \end{aligned}$$

$\square$

Corollary III.3.11. *Let $x = L(y_0 \circ_1 \ldots \circ_1 y_m) \in Q^*$. Then,*

$$\mathsf{Op}^{+n}_x \Leftrightarrow \mathsf{Op}^{+(n+m)}_{y_0} \circ \mathsf{Op}^{+(n+m-1)}_{y_1} \circ \ldots \circ \mathsf{Op}^{+n}_{y_m}.$$

Proof By Lemma III.1.10 and Lemma III.3.10 $\square$

We conclude by looking at some examples of operations.

Exercise III.3.12. *Consider the names $x' := (2,(3,(4,q_0)))$, $y' := (\gamma,(\gamma_1,(\gamma_2,q_0)))$, $x := (2,(3,(4,q_0)))$, $y := (2,(3,(\gamma,q_0)^-))$ and $z := (2,(3,(\gamma,q_0)^-)^-)$. Compute $\mathsf{Op}_{x'}$, $\mathsf{Op}_{y'}$, Op_x, Op_y and Op_z.*

Solution III.3.12. *Below, we just give the solutions to (i)–(iv), (v) is explained in detail. To obtain these solutions, we have used that $(\forall\xi<\gamma)\mathsf{p}_n\mathsf{p}_n^{\xi}\mathsf{p}_n \Leftrightarrow \mathsf{p}_n^{\gamma}$, that $\mathsf{p}_3^{\gamma}\mathsf{p}_1$ iff $\mathsf{p}_3^{\gamma}\mathsf{p}_2^{n}\mathsf{p}_1$.*

(i) $\mathsf{Op}_{x'} \Leftrightarrow ((\mathsf{p}_3^4\mathsf{p}_2)^3)\mathsf{p}_1)^2$.

(ii) $\mathsf{Op}_{y'} \Leftrightarrow (\mathsf{p}_1(\mathsf{p}_2\mathsf{p}_3^{\gamma_2})^{\gamma_1}\mathsf{p}_2)^{\gamma}\mathsf{p}_1$.

(iii) $\mathsf{Op}_{x} \Leftrightarrow ((\mathsf{p}_3^{\gamma}\mathsf{p}_2)^3\mathsf{p}_1)^2$.

(iv) $\mathsf{Op}_{y} \Leftrightarrow [(\forall\xi<\gamma)(\mathsf{p}_2\mathsf{p}_3^{1+\xi}\mathsf{p}_2)^3\mathsf{p}_1]^2$.

(v) $\mathsf{Op}_{z} \Leftrightarrow [((\forall\eta<\gamma)(\mathsf{p}_1\mathsf{p}_3^{1+\eta})\mathsf{p}_2 \circ ((\forall\xi<\gamma)(\mathsf{p}_2\mathsf{p}_3^{1+\xi})\mathsf{p}_2)^2)\mathsf{p}_1]^2$.

We details for (v). Let $y := (3,(\gamma,q_0)^-)$ and $x := (\gamma,q_0)$. Then, $z = (2,y^-)$ and $y = (3,x^-)$. Thus, $\mathsf{Op}_z \Leftrightarrow \mathsf{Op}_{(2,y^-)} \Leftrightarrow (\mathsf{Op}_{y^-}^{+(0,1)}\mathsf{p}_1)^2$. By definition,

a) $\mathsf{Op}_{y^-}^{+(0,1)} \Leftrightarrow \mathsf{Op}_{(1,x^-)^-}^{+(0,1)} \circ \mathsf{Op}_{(2,x^-)}^{+1}$,

b) $\mathsf{Op}_{(1,x^-)}^{+(0,1)} \Leftrightarrow \mathsf{Op}_{x^-}^{+(0,2)}\mathsf{p}_2 \Leftrightarrow \mathsf{Op}_{x^-}^{+(0,2)}\mathsf{p}_2$, *and* $\mathsf{Op}_{(2,x^-)}^{+1} \Leftrightarrow (\mathsf{Op}_{x^-}^{+(1,2)}\mathsf{p}_2)^2$,

c) $\mathsf{Op}_{x^-}^{+(0,2)} \Leftrightarrow (\forall\eta<\gamma)(\mathsf{p}_1\mathsf{p}_3^{1+\eta})$ *and* $\mathsf{Op}_{x^-}^{+(1,2)} \Leftrightarrow (\forall\xi<\gamma)(\mathsf{p}_2\mathsf{p}_3^{1+\xi})$.

Putting the pieces together, we obtain that

$$\mathsf{Op}_{y^-}^{+(0,1)} \Leftrightarrow (\forall\eta<\gamma)(\mathsf{p}_1\mathsf{p}_3^{1+\eta})\mathsf{p}_2 \circ ((\forall\xi<\gamma)(\mathsf{p}_2\mathsf{p}_3^{1+\xi})\mathsf{p}_2)^2,$$

which confirms (v).

III.4 Approximations and normal forms

In this section, we have a closer look at the names in Q, which we use to name operations. First, we lift the notions degree, ordinal, normal forms and approximations defined for names in Q_2 in Section II.2 to Q. We will see that all relevant properties are preserved. For instance, we will have again for each name x with $\deg(x) = m+1$, $\breve{\mathsf{T}}_x$ iff $(\forall\alpha \lhd o(x))(\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{x[\alpha]}))$ (cf. Lemma III.6.2). Then, we lift the well-founded relations $\rightsquigarrow$ and $\rightsquigarrow^*$ from Q_2 to Q, and the map $H : Q_2 \to Q_2^H$ to $H : Q \to Q^H$, so that Op_x corresponds to H_{x^H}.

Often, the definitions look exactly the same as in the case $N_0 = 2$. However, as the underlying names $(Q,<)$ are different, properties have to rechecked.

Definition III.4.1. *For $x \in Q$ and f denoting one of the function symbols in $\{\deg, o\}$, we let $f(x) := f((x)_0)$ and $f(\alpha{+}1, v) := f(1, v)$. Further,*

(i) $\deg(q_0) := 0$, $\deg(1, x^-) := 1$, $\deg(1, x) := \deg(x){+}1$ *and* $\deg(\gamma, v) := 1$.

(ii) $o(q_0) := 1$, $o(1, x^-) := o(x)$, $o(1, x) := o(x)$ *and* $o(\gamma, v) := \gamma$.

We extend $\deg$ *and* o *to* $Q \cup P$ *by setting,* $\deg(x^-) := 0$ *and* $o(x^-) := o(x)$*. Further, we read a name* $(0, v)$ *as an abbreviation for* q_0*.*

As with names and prenames in $Q_2 \cup P_1$, if $v \in Q \cup P$, then $v{+}1$ denotes its successor w.r.t. the ordering $(Q \cup P, <)$. Again, $x{+}1 := q_1 \circ x$, if $x^- \in P$, then $x^-{+}1 := x$, and $\langle x_1, \ldots, x_k\rangle{+}1 = \langle x_1{+}1, \ldots, x_k\rangle$.

We start with some simple properties of $\deg(x)$ and $o(x)$.

Lemma III.4.2.

(i) $o(x{+}1) = 1$, *and* $x^- \in P$ *iff* $o(x) \in \mathsf{Lim}(\Omega) \wedge \deg(x) = 1$.

(ii) If $x \circ y \in Q$ *and* $x \neq q_0$, *then* $\deg(x) = \deg(x \circ y)$, *and* $o(x) = 1$ *iff* $o(x \circ y) = 1$, *and* $o(x) \in \mathsf{Lim}(\Omega)$ *iff* $o(x \circ y) \in \mathsf{Lim}(\Omega)$.

(iii) If $x = L(x_0 \circ_1 (1, y_1) \circ_1 \ldots \circ_1 (1, y_m))$ *and* $x_0 \neq q_0$, *then* $\deg(x) = \deg(x_0){+}m$, *and* $o(x_0) = 1$ *iff* $o(x) = 1$, *and* $o(x_0) \in \mathsf{Lim}(\Omega)$ *iff* $o(x) \in \mathsf{Lim}(\Omega)$.

Proof (i) $x{+}1$ is of the form $(\alpha{+}1, q_0) * z$, hence $o(x) = o(\alpha{+}1, q_0) = o(1, q_0) = 1$. The second part is by induction on $\mathsf{lv}(x)$. If $\mathsf{lv}(x) = 1$, the $x^- \in P$ iff $x = (\gamma, q_0)$. And if $\mathsf{lv}(x) > 1$, then $x^- \in P$ if either $(x)_0 = (\gamma, v)$ or $(x)_0 = (\beta{+}1, y^-)$ with $\mathsf{lv}(y) < \mathsf{lv}(x)$, so $o(x) = o(y) \in_{IH} \mathsf{Lim}(\Omega)$. Hence in both cases $o(x) \in \mathsf{Lim}(\Omega)$ and $\deg(x) = 1$. And if $o(x) \in \mathsf{Lim}(\Omega) \wedge \deg(x) = 1$, then also $(x)_0 = (\gamma, v)$ or $(x)_0 = (\beta{+}1, y^-)$, and $x^- \in P$ by definition of P. (ii) If $x \circ y = x * y$ or $\mathsf{lh}(x) > 1$, then the claim is directly by the definition of deg and o. Otherwise, $x = (\alpha, v)$ and $y = (\beta, v) * y'$, and $\deg(x \circ y) = \deg(\beta{+}\alpha, v)$, and $o(x \circ y) = o(\beta{+}\alpha, v)$. Since $\beta{+}\alpha \in \mathsf{lim}(\Omega)$ iff $\alpha \in \mathsf{lim}(\Omega)$, the claim follows. (iii) By induction on m: for $m = 0$, there is nothing to show, and if $m = m'{+}1$, and say $x_m = (1, y_m)$, then, for $z := L(x_0 \circ_1 \ldots \circ_1 x_{m'})$, $x = z \circ_1 (1, y_m)$, and $\deg(x) = \deg(z \circ_1 (1, y_m)) = \deg(z \circ y_m)){+}1 = \deg(z){+}1 =_{IH} \deg(x_0){+}m'{+}1$, and $o(x) = o(z \circ y_m)$, and the claim is by (ii) and the I.H. □

We consider $L(x_0 \circ_1 (1, y_1) \ldots \circ_1 (1, y_m)) \circ y_{m+1}$ an expression in normal form, if x_0 is either q_1, or of the form $(1, z^-)$ or (γ, v). In the third case, we want that γ is largest possible: $(\omega{+}\omega, q_0) \circ_1 (1, q_0)$ is in normal form, but $(\omega, q_0) \circ_1 (1, (\omega, q_0))$ is not. This is why we additionally ask for $x_0 * y_1 \in Q$ in this case.

Definition III.4.3. *$L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1} \in Q$ is an expression in (long) normal form, if either $m = 0$ and $x_0 = q_1 \vee x_0 = (1, z^-) \vee (x_0 = (\gamma, v) \wedge x_0 * y_1 \in Q)$, or $m > 1$ and $x_1 = (1, y_1), \ldots, x_m = (1, y_m)$ and*

(i) $x_0 = q_1$ or $x_0 = (1, z^-)$, or

(ii) $x_0 = (\gamma, v)$ and $x_0 * y_1 \in Q$.

We write $z =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$ if $z = L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$ and $L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$ is an expression in normal form.

When we introduced normal forms in the previous chapter, we did not yet have the partial function $\circ_1$ at hand. Thus, we considered $(1, (\gamma, q_0))$ and $(1, (\alpha{+}1, q_0))$ as normal forms. Now, we have $(1, (\gamma, q_0)) =_{NF} (\gamma, q_0) \circ_1 (1, q_0)$ and $(1, (\alpha{+}1, q_0)) =_{NF} q_1 \circ_1 (1, (\alpha, q_0))$.

Each name $x \in Q^*$ has a unique normal form.

Lemma III.4.4. *If $\deg(x) = m{+}1$, then there are unique $x_0, \ldots, x_{m+1} \in Q$, so that $x =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$.*

Proof By induction on m. If $\deg(x) = 1$, then either $(x)_0 = (\alpha{+}1, q_0) = q_1 \circ (\alpha, q_0)$, $(x)_0 = (1{+}\beta, y_0^-)$, or $(x)_0 = (\gamma, y_0)$. Therefore, if $x = (x)_0 * z$, then either $x = q_1 \circ y_1$ for $y_1 := (\alpha, q_0) * z$, or $x = (1, y_0^-) \circ y_1$ for $y_1 := (\beta, y_0^-) * z$, or $x = (\gamma, y_0) * y_1$ for $y_1 := z$. Further, these representations are unique.

If $\deg(x) = m{+}2$ and $x = (x)_0 * z$, then $(x)_0$ is of the form $(\alpha{+}1, y)$, and $x = (1, y) \circ y_{m+2}$ for $y_{m+2} := (\alpha, y) * z$, where $\deg(y) = m{+}1$. Note that y and y_{m+2} are uniquely determined. By I.H., we have $y =_{NF} L(x_0 \circ_1 x_1 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$. Then, $(1, y) = (1, L(x_0 \circ_1 x_1 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}) = L(x_0 \circ_1 \ldots \circ_1 x_m) \circ_1 (1, y_{m+1})$, and $x =_{NF} (L(x_0 \circ_1 \ldots \circ_1 x_m) \circ_1 (1, y_{m+1})) \circ y_{m+2}$. The conditions on x_0 and x_1 are also immediate by the I.H. □

The following observations allow us to define short normal forms.

Lemma III.4.5. *Let $x =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$ and $k \leq m$.*

(i) $z_k := L(x_0 \circ_1 \ldots \circ_1 x_k)$ is in normal form and $\deg(z_k) := k{+}1$.

(ii) If $0 < n \leq m$, then $x = L(x_0 \circ_1 \ldots \circ_1 x_{m-n}) \circ_n L(x_{m-n+1} \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$.

Proof (i) By Definition III.4.3 and Lemma III.4.2 (iii). (ii) By Corollary III.1.11. □

Definition III.4.6. *Let $x =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$ (so $\deg(x) = m{+}1$). Then, we write for each $n < \deg(x)$,*

$$x =_{NF} y \circ_n z,$$

if $y = L(x_0 \circ_1 \ldots \circ_1 x_{m-n})$ and $z = L(x_{m-n+1} \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$. We then call $y \circ_n z$ the short normal form of x. Further, we call y a simple name of degree $m{-}n{+}1$.

For the following discussion let $x =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ x_{m+1}$, so $\deg(x) = m{+}1$. Note that $x =_{NF} y \circ_0 z$ is always available; then $y = L(x_0 \circ_1 \ldots \circ_1 x_m)$ and $z = x_{m+1}$. If $\deg(x) > 1$, then we most often use the short normal form $x =_{NF} y \circ_{m-1} z$; in this case, $y = L(x_0 \circ_1 x_1)$ is a simple name of degree two, and $z = L(x_2 \circ_1 \ldots \circ_1 x_m) \circ y_{m+1}$. Further, if $x =_{NF} y \circ_m z$, then $y = x_0$ is a simple name of degree one.

Lemma III.4.7. *If $\deg(x) = m{+}1$ and $n \leq m$, then there exists unique names $y, z \in Q$, y simple with $\deg(y) = m{-}n{+}1$, so that $x =_{NF} y \circ_n z$.*

Proof By Definition of the short normal form and since the (long) normal form is unique. □

The following helps to find short normal forms.

Lemma III.4.8. *Let $x \in Q^*$.*

(i) If $x =_{NF} y \circ_n z$, then $x \circ z' =_{NF} y \circ_n (z \circ z')$.

(ii) If $x =_{NF} y \circ_n z$, then $(1, x) =_{NF} y \circ_{n+1} (1, z)$.

(iii) If $(1, v) =_{NF} y \circ_n z$, then $(\beta{+}1, v) =_{NF} y \circ_n ((\beta, v) \circ z)$.

Proof Let $x =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m) \circ x_{m+1}$. Straightforward computation verifies the claims. (i) Note that $x \circ z' = L(x_0 \circ_1 \ldots \circ_1 x_m) \circ (x_{m+1} \circ z')$ by Lemma III.1.8. (ii) We have $(1, x) =_{NF} L(x_0 \circ_1 \ldots \circ_1 x_m \circ_1 (1, x_{m+1}))$, and $(1, x) = (1, y \circ_m z) = y \circ_{m+1} (1, z)$, and $z = L(x_{m-n+1} \circ_1 \ldots \circ_1 x_m) \circ x_{m+1}$. Thus, $(1, z) = L(x_{m-n+1} \circ_1 \ldots \circ_1 x_m \circ_1 (1, x_{m+1}))$. (iii) $(\beta{+}1, v) = (1, v) \circ (\beta, v)$, so the claim is by (i). □

Example III.4.9. *Consider the name $x = (1, (1, y))$, where $y := \langle (1, q_1), (1, q_1^2) \rangle$. We have $H_y = \mathsf{It}(\mathsf{it}) \circ \mathsf{It}^2(\mathsf{it})$ and Op_y iff $(\mathsf{p}_2\mathsf{p}_1)(\mathsf{p}_2^2\mathsf{p}_1)$, and*

(i) $H_x = H_y^{+2}[\mathsf{It}, \mathsf{it}] = (\mathsf{It}_4(\mathsf{It}_3) \circ \mathsf{It}_4^2(\mathsf{It}_3), \mathsf{It}, \mathsf{it})$.

(ii) Op_x iff $\mathsf{Op}_{(1,y)}^{+}\mathsf{p}_1$ iff $\mathsf{Op}_y^{+2}\mathsf{p}_2\mathsf{p}_1$ iff $\mathsf{Op}_y^{+2}\mathsf{p}_1$ iff $(\mathsf{p}_4\mathsf{p}_3)(\mathsf{p}_4^2\mathsf{p}_1)$.

It is readily checked that the (long) normal form and the short normal form look as follows:

$$x =_{NF} L(q_1 \circ_1 q_1 \circ_1 (1, z) \circ_1 q_1) =_{NF} q_2 \circ_2 (1, (1, z)), \text{ where } z := (1, q_1^2).$$

Next, we have a glance at what will be instances of the Lemmas III.2.9, III.6.11 and III.6.10. These results state that the (long) normal form goes well with functionals and operations, and the short normal form goes well with operations in the following sense:

(i) $H_x = (H_{q_1}^{+3}, H_{q_1}^{+2}, H_{(1,z)}^{+}, H_{q_1})$.

(ii) Op_x *iff* $\mathsf{Op}^{+3}_{q_1} \circ \mathsf{Op}^{+2}_{q_1} \circ \mathsf{Op}^{+}_{(1,z)} \circ \mathsf{Op}_{q_1}$.

(iii) Op_x *iff* $\mathsf{Op}^{+2}_{q_2} \circ \mathsf{Op}_{(1,(1,z))}$.

Let us verify these claims. First, observe that $H_z = (\mathsf{It}^2, \mathsf{it})$ *and* $H_{(1,z)} = (\mathsf{It}^2_3, \mathsf{It}, \mathsf{it})$, Op_z *iff* $\mathsf{p}^2_2\mathsf{p}_1$ *and* $\mathsf{Op}_{(1,z)}$ *iff* $\mathsf{p}^2_3\mathsf{p}_1$. *So* $H_{(1,(1,z))} = (\mathsf{It}^2_4, \mathsf{It}_3, \mathsf{It}, \mathsf{it})$, *and* $\mathsf{Op}_{(1,(1,z))}$ *iff* $\mathsf{p}^2_4\mathsf{p}_1$.

(i) Indeed we have that

$$(H^{+3}_{q_1}, H^{+2}_{q_1}, H^{+}_{z}, H_{q_1}) = (\mathsf{It}_4, \mathsf{It}_3, (\mathsf{It}^2_4, \mathsf{It}_3, \mathsf{It}), \mathsf{it}) =$$
$$(\mathsf{It}_4(\mathsf{It}_3), (\mathsf{It}^2_4(\mathsf{It}_3), \mathsf{It}), \mathsf{it}) = (\mathsf{It}_4(\mathsf{It}_3) \circ \mathsf{It}^2_4(\mathsf{It}_3), \mathsf{It}, \mathsf{it}) = H_x.$$

(ii) $\mathsf{Op}^{+3}_{q_1} \circ \mathsf{Op}^{+2}_{q_1} \circ \mathsf{Op}^{+}_{(1,z)} \circ \mathsf{Op}_{q_1}$ *iff* $\mathsf{p}_4 \circ \mathsf{p}_3 \circ (\mathsf{p}^2_4\mathsf{p}_2) \circ \mathsf{p}_1$ *iff* $(\mathsf{p}_4\mathsf{p}_3)(\mathsf{p}^2_4\mathsf{p}_1)$ *iff* Op_x.

(iii) $\mathsf{Op}^{+2}_{q_2} \circ \mathsf{Op}_{(1,(1,z))}$ *iff* $(\mathsf{p}_4\mathsf{p}_3) \circ (\mathsf{p}^2_4\mathsf{p}_1)$ *iff* Op_x.

With functionals, the short normal form is not very helpful. H_x *is not easily reconstructed from* $H^{+2}_{q_2} = \mathsf{It}_4(\mathsf{It}_3)$ *and* $(\mathsf{It}^2_4, \mathsf{It}_3, \mathsf{It}, \mathsf{it})$, *as one would have to look inside the type-*2 *functional* $(\mathsf{It}^2_4, \mathsf{It}_3, \mathsf{It}, \mathsf{it})$.

We use the (unique) normal form of a name $x \in Q^*$ to define an approximation $x[\alpha]$, and if $\deg(x) > 1$ another approximation $x(\alpha)$. The definition is by recursion on the level. For clarity, we list the case where $\deg(x) = 1$ separately.

Definition III.4.10. *Let* $x \in Q^*$. *If* $\deg(x) = 1$, *then either*

(i) $x =_{NF} q_1 \circ z$ *and* $x[\alpha] := z$,

(ii) $x =_{NF} (\gamma, v) \circ z$ *and* $x[\alpha] := (1+\alpha, v) \circ z$ *if* $\alpha < \gamma$, *and else* $x[\alpha] := x$,

(iii) $x =_{NF} (1, y^-) \circ z$ *and* $x[\alpha] := (1, y[\alpha]) \circ z$.

And if $\deg(x) = m > 1$, *then we have one of the following cases.*

(i) $x =_{NF} L(q_1 \circ_1 (1, y_2) \circ_1 \ldots \circ_1 (1, y_m)) \circ z$, $x[\alpha] := L((1+\alpha, y_2) \circ_1 \ldots \circ_1 (1, y_m)) \circ z$, *and* $x(\alpha) := x[\alpha]+1$,

(ii) $x =_{NF} L((\gamma, y_1) \circ_1 (1, y_2) \ldots \circ_1 (1, y_m)) \circ z$, $x[\alpha] := L((\gamma, y_1)[\alpha] \circ_1 \ldots \circ_1 (1, y_m)) \circ z$, *and* $x(\alpha) := L((1+\alpha, w) \circ_1 (1, y_3) \circ_1 \ldots \circ_1 (1, y_m)) \circ z$ *for* $w := ((\gamma, y_1) * y_2)^-$.

(iii) $x =_{NF} L((1, y_1^-) \circ_1 (1, y_2) \ldots \circ_1 (1, y_m)) \circ z$, $x[\alpha] := L((1, y_1[\alpha]) \circ_1 \ldots \circ_1 (1, y_m)) \circ z$, *and* $x(\alpha) := L((1+\alpha, w) \circ_1 (1, y_3) \circ_1 \ldots \circ_1 (1, y_m)) \circ z$ *for* $w := ((1, y_1^-) * y_2)^-$.

Below, we list some first properties of these approximations.

Lemma III.4.11. *Let $x \in Q^*$. Then we have the following.*

(i) If $x =_{NF} y \circ_m z$, then $x[\alpha] = y[\alpha] \circ_m z$, and if further $\deg(y) > 1$, then $x(\alpha) = y(\alpha) \circ_m z$.

(ii) If $\deg(x) = m{+}2$, then $\deg(x(\alpha)) = m{+}1$, and if further $o(x) = 1$, then $x(\alpha) = q_1 \circ_m x[\alpha]$.

Proof (i) By Lemma III.4.5 (iii). (ii) The first part is immediate by Lemma III.4.5 (iii). For the second part note that if $\deg(x) = m{+}2$ and $o(x) = 1$, then $x =_{NF} (1, y{+}1) \circ_m z$, and $x[\alpha] = (1, y) \circ_m z$ and $x(\alpha) = (q_1 \circ (1, y)) \circ_m z =_{L.III.1.8}= q_1 \circ_m ((1, y) \circ_m z) = x[\alpha]$. $\square$

When dealing with approximations, also the following simple properties are useful.

Lemma III.4.12. *For each $x \in Q^*$ and all $v \in Q^* \cup P$,*

(i) $(x{+}1)[\alpha] = x$, $(1, x{+}1)[\alpha] = (1{+}\alpha, x)$ and $(1, v{+}1)(\alpha) = (1{+}\alpha, v)$.

(ii) if $x = (x)_0 * y$, then $x[\alpha] := (x)_0[\alpha] * y$ and if $\deg(x) > 1$, $x(\alpha) := (x)_0(\alpha) * y$,

(iii) $(\beta{+}1, v)[\alpha] := (1, v)[\alpha] \circ (\beta, v)$, and $(\beta{+}1, z)(\alpha) := (1, z)(\alpha) \circ (\beta, y)$.

(iv) $(1, v)[\alpha] := (1, v[\alpha])$ if $v \neq y{+}1$ (for some $y \in Q$).

Proof (i)-(iii) is immediate by Definition III.4.6 and Lemma III.4.8. (iv) If $v = z^-$, this is by Definition III.4.10. Else, $x = v \in Q$. Say, $x =_{NF} y \circ z$. By Lemma III.4.11 (ii), $x[\alpha] =_{NF} y[\alpha] \circ z$. Then, $(1, v) =_{NF} y \circ_1 (1, z)$, and $(1, v)[\alpha] = y[\alpha] \circ_1 (1, z) = (1, y[\alpha] \circ z) = (1, x[\alpha]) = (1, v[\alpha])$. $\square$

If $\deg(x) = m{+}2$, then $x =_{NF} y \circ_m z$ for some simple name of degree two, and $x[\alpha] = y[\alpha] \circ_m z$ and $x(\alpha) = y(\alpha) \circ_m z$. Therefore, we observe the following.

Lemma III.4.13. *Assume that x is a simple name with $\deg(x) = m{+}2$. Then,*

(i) If $o(x) \in \mathsf{Lim}(\Omega)$, then $\deg(x(\alpha)) = m{+}1$, $o(x(0)) = o(x)$, $x(0)[\alpha] = x[\alpha]$, $o(x(\gamma)) = \gamma$ and $x(\gamma)[\alpha] = x(\alpha)$ for $\alpha < \gamma$. Further, $x(0) \circ x(\alpha) = x(\alpha{+}1)$.

(ii) If $o(x) = 1$, then $\deg(x[\gamma]) = 1$, $o(x[\gamma]) = \gamma$ and $x[\gamma][\alpha] = x[\alpha]$ for $\alpha < \gamma$. Further, $x[0] \circ x[\alpha] = x[\alpha{+}1]$.

Also the following technical results are obtained completely analogously to the case $N_0 = 2$.

Lemma III.4.14. *If $\deg(x) = 1$ and $o(x) = \gamma$, then $x = \mathsf{sup}_{<}\{x[\alpha] : \alpha < \gamma\}$.*

Lemma III.4.15. *If $x \in Q$ with $\deg(x) > 1$, and $q_0 \neq y \in Q^H$ is not a successor, then* *(i)* *if* $o(x) = 1$*, then* $x = \mathsf{sup}_<\{x[\alpha] < x : \alpha < \Omega\} = \mathsf{sup}_<\{x(\alpha) : \alpha < \Omega\}$*,*

(ii) *if* $o(x) = \gamma$*, then* $x(0) = \mathsf{sup}_<\{x[\alpha] : \alpha < \gamma\}$*, and* $x = \mathsf{sup}_<\{x(\alpha) : \alpha < \Omega\}$*,*

(iii) $y = \mathsf{sup}_{<^H}\{y[\alpha] < y : \alpha < \Omega\}$*.*

Lemma III.4.16. *Assume that* $x \in Q^*$ *and* $x \circ y \in Q^*$*, and let*

$$\delta_0 := \begin{cases} \delta : & x = (\gamma, v) \wedge (y)_0 = (\delta, v), \\ 0 : & \textit{otherwise.} \end{cases}$$

Then, for each α*,* $x[\alpha] \circ y = (x \circ y)[\delta_0 + \alpha]$*, and* $o(x \circ y) = \delta_0 + o(x)$*.*

Lemma III.4.17. *Assume that* $x \in Q^*$ *with* $\deg(x) > 1$*. If* $x \circ y \in Q^*$*, then we have that* $x(\alpha) \circ y = (x \circ y)(\alpha)$ *and* $o(x \circ y) = o(x)$*.*

Finally, we extend the relations $\rightsquigarrow$ and $\rightsquigarrow^*$ to Q. This is straightforward.

Definition III.4.18. *All all* $x, y \in Q$*,* $y \rightsquigarrow x :\Leftrightarrow (\exists \alpha < o(x))(y = x[\alpha])$*. Further,* $\rightsquigarrow^*$ *is the transitive closure of* $\rightsquigarrow$*, and* $\rightsquigarrow^*_r$ *is reflexive closure of* $\rightsquigarrow^*$*.*

Lemma III.4.19.

(i) $(Q, \rightsquigarrow^*)$ *is well-founded.*

(ii) *If* $q_0 < x \in Q$*, then* $q_0 \rightsquigarrow^* x$*.*

(iii) *If* $y \rightsquigarrow^* x$*, then either* $y \rightsquigarrow x$ *or* $y \rightsquigarrow^* x[\alpha] \rightsquigarrow x$ *for some* $\alpha < o(x)$*.*

(iv) *If* $y * z \in Q$ *and* $x * z \in Q$*, then* $y \rightsquigarrow^* x \Rightarrow y * z \rightsquigarrow^* x * z$*.*

(v) *If* $1 \leq \alpha < \beta$*, then* $(\alpha, v) \rightsquigarrow^* (\beta, v)$*.*

(vi) *If* $y \rightsquigarrow^* x$*, then* $(1, y) \rightsquigarrow^* (1, x)$*.*

(vii) *If* $\alpha < \beta < o(x) = \gamma$*, then* $x[\alpha] \rightsquigarrow^* x[\beta]$*.*

(viii) *If* $y \rightsquigarrow^* x$ *and* $z \rightsquigarrow^* x$*, then* $y \rightsquigarrow^* z \vee y = z \vee z \rightsquigarrow^* y$*.*

(ix) *If* $y \circ_m z \in Q$ *and* $x \circ_m z \in Q$*, then* $y \rightsquigarrow^* x \Rightarrow y \circ_m z \rightsquigarrow^* x \circ z$*.*

(x) $q_m \rightsquigarrow^* q_{m+1}$*, and if* $\deg(x) = m+1$*, then* $q_{m+1} \rightsquigarrow^*_r x$ *(where* $q_{m+1} := (1, q_m)$*),*

(xi) *If* $\deg(x) > 1$ *and* $\beta < \alpha$*, then* $x(\beta) \rightsquigarrow^* x(\alpha)$*.*

(xii) $\mathsf{Wo}_{\rightsquigarrow^*}(x)$*.*

Proof We just show (ix) and (x), as the other claims are shown very similar to the corresponding claims of Lemma II.2.16. (ix) By induction on m. The case $m = 0$ is shown as the corresponding claims of Lemma II.2.16. To show the induction step, assume $y \rightsquigarrow^* x$, $x \circ_{m+1} z \in Q$ and that the claim holds for m. As $x \circ_{m+1} z$ is defined, $z = (1, z') * z''$. By I.H., $y \circ_m z' \rightsquigarrow^* x \circ_m z'$, thus $(1, y \circ_m z') \rightsquigarrow^* (1, x \circ_m z')$ by (vi), and the claim follows by the case $m = 0$. (x) By I.H. on m, one readily obtains that $q_{m+1}[0] = q_m$. If $\deg(x) = m{+}1$, then $x =_{NF} L(x_1 \circ_1 \ldots \circ_1 x_{m+1}) \circ x_{m+2}$. As $q_{m+1} = L(q_1 \circ_1 \ldots \circ_1 q_1)$, the claim is by (ix). (xi) By Lemma III.4.11 and (ix), it suffices to show the claim for $\deg(x) = 2$, which is done analogously to the corresponding case of Lemma II.2.16. $\square$

As in the previous chapter, the map $\cdot^H : Q \to Q^H$ assigns to each $x \in Q$ a name $x^H \in Q^H$, so that Op_x corresponds to H_{x^H}.

Definition III.4.20. *We define* $\mathsf{corr} : Q \to \{0, 1\}$ *and* $\cdot^H : Q \to Q^H$ *as follows.*

(i) $\mathsf{corr}(x) := 1$ *if* $\exists y, n[x = y{+}n \wedge \deg(y) = 1 \wedge o(y) \in \mathsf{Lim}(\Omega)]$*; else* $\mathsf{corr}(x) := 0$.

(ii) $(q_0)^H := q_0$, $(\alpha, y^-)^H := (\alpha, y^H)$ *and* $(\alpha, y)^H := (\alpha, y^H{+}\mathsf{corr}(y))$,

(iii) if $k > 1$, *then* $\langle x_1, \ldots, x_k\rangle^H := \langle x_1^H, \ldots, x_k^H\rangle$.

We also write $H(x)$ *for* x^H.

Note that $\mathsf{corr}(x) = \mathsf{corr}(x{+}1)$ and that $(x{+}1)^H = x^H{+}1$. To avoid case distinctions, we extend corr by letting $\mathsf{corr}(x^-) := 1$. Note that $(\alpha, v)^H = (\alpha, (v{+}\mathsf{corr}(v))^H)$: $(\alpha, y^-)^H = (\alpha, y^H) = (\alpha, (y^-{+}1)^H))$, and further, $(\alpha, y)^H = (\alpha, (y^H{+}\mathsf{corr}(y)) = (\alpha, (y{+}\mathsf{corr}(y))^H)$.

Below, we verify the indeed $\cdot^H : Q \to Q^H$.

Lemma III.4.21. *For all* n *and all* $x \in Q_n$, $x^H \in Q_n^H$ *and* $x < y \Rightarrow x^H \leq y^H$.

Proof We show the claim by induction on n. If $x, y \in Q_1$, then both claims are obvious. Now assume that both claims hold for $n > 0$, and let $x = \langle(\alpha_1, v_1), \ldots, (\alpha_k, v_k)\rangle$ and $y = \langle(\beta_1, w_1), \ldots, (\beta_l, w_l)\rangle$ with $x, y \in Q_{n+1}$ and $x < y$. We show that $x^H \in Q_{n+1}^H$ and $x^H < y^H$. Note that if $x^- \in P$, then by Lemma III.4.2, $\deg(x) = 1$ and $o(x) \in \mathsf{Lim}(\Omega)$, therefore $(\omega, q_0) \leq x$ and $\mathsf{corr}(x) = 1$. Further, $\mathsf{corr}(x) = \mathsf{corr}(x{+}1)$. Hence, if $v, w \in Q_n \cup P_n$ with $v < w$, then $v{+}\mathsf{corr}(v) < w{+}\mathsf{corr}(w)$. As $v{+}\mathsf{corr}(v), w{+}\mathsf{corr}(w) \in Q_n$, the I.H. yields $(v{+}\mathsf{corr}(v))^H < (w{+}\mathsf{corr}(w))^H$. By the initial remark, $(\alpha_i, v_i)^H = (\alpha_i, (v_i{+}\mathsf{corr}(v_i))^H$. Thus, $x^H \in Q_{n+1}^H$. And as $x < y$, either x is either an initial segment of y, and then x^H is an initial segment of y^H, or there is a first position form the right where x and y differ, say $(\alpha_i, v_i) < (\beta_j, w_j)$. By the above, also $(\alpha_i, v_i)^H < (\beta_j, w_j)^H$, which is now the first position form the right where x^H and y^H differ, thus $x^H <_{n+1} y^H$. $\square$

Lemma III.4.22. *$(x \circ y)^H = x^H \circ y^H$.*

Proof By definition, $(x * y)^H = x^H * y^H$. As further, for $z := v+\mathsf{corr}(v)$, we have $((\alpha, v) \circ (\beta, v))^H = (\beta+\alpha, z^H) = (\alpha, z^H) \circ (\beta, z^H) = (\alpha, v)^H \circ (\beta, v)^H$, the claim follows. □

Lemma III.4.23.

(i) $\deg(x) \leq \deg(x^H)$,

(ii) if $\deg(x) = 1$, *then* $o(x) = o(x^H)$,

(iii) if $\deg(x) > 1$, *then* $o(x^H) = 1$.

Proof All claims are shown by induction on $\mathsf{lv}(x)$. (i) As $x^H = q_0$ implies $x = q_0$, the claim holds if $\deg(x) \leq 1$. And if $\deg(x) > 1$, we have that $x = (\alpha+1, y) * z$, and so $\deg(x) = \deg(y)+1$ and $\deg(x^H) = \deg(y^H+\mathsf{corr}(y))+1$. If $\deg(y) = 1$, then the claim holds as $\deg(y^H+\mathsf{corr}(y)) \geq 1$, and if $\deg(y) > 1$, then $\mathsf{corr}(y) = 0$, and $\deg(y) \leq \deg(y^H)$ by I.H., so $\deg(x) \leq \deg(x^H)$.
(ii) If $\deg(x) = 1$, x is of the form $y_0 := (1, q_0) \circ z$ or $y_1 := (\gamma, v) \circ z$ or $y_2 := (1, z^-) \circ z'$. Using Lemma III.4.22, we see that $o(y_0) = o(y_0^H) = 1$ and $o(y_1) = o(y_1^H) = \gamma$. With y_2, note that $z \in P$. By Lemma III.4.2, $\deg(z) = 1$, so $o(y_2) = o(z) =_{IH} o(z^H) = o(y_2^H)$.
(iii) If $\deg(x) = 2$, it suffices, by the above Lemma, to check the claim for simple names of degree two, names of the form $(1, y+1)$, $(1, (\gamma, v))$ and $(1, (\beta+1, y^-))$, which is straightforward. And if $\deg(x) > 2$, then x is of the form $(1, y) \circ z$ for $\deg(y) \geq 2$, then the claim is by I.H. □

Finally, we lift Lemma II.2.19.

Lemma III.4.24. *Let $x \in Q$.*

(i) If $\deg(x) = 1$ *and* $o(x) = \gamma$, *then* $x^H[\alpha] \rightsquigarrow_r^* (x[\alpha])^H \rightsquigarrow_r^* x^H[\alpha+1]$.

(ii) If $\deg(x) = m+2$, *then* $x^H[\alpha] \rightsquigarrow_r^* (x(\alpha))^H \rightsquigarrow_r^* x^H[\alpha+1]$.

Proof Again, we just show the first claim of (ii). Let $x =_{NF} y \circ_m z$. Then y is a simple name of degree two, and by Lemma III.4.22 $x^H[\alpha] = y^H[\alpha] \circ_m z$ and $H(x(\alpha)) = H(y(\alpha)) \circ_m z^H$. By Lemma III.4.19 (ix), it is thus enough to check the claim for simple names of degree two. To do so, recall for each y, $(y+1)^H = y^H+1$ and $\mathsf{corr}(y) = \mathsf{corr}(y) + 1$. Hence, $(1, y+1)^H[\alpha] = (1, y^H+\mathsf{corr}(y)+1)[\alpha] = (1+\alpha, y^H+\mathsf{corr}(y)) = (1+\alpha, y)^H = H(x[\alpha])$, and further, for for $y = (\gamma, v)$ and $y = (\beta+1, z^-)$, $\mathsf{corr}(y) = 1$, $y^- \in P$, $(1+\alpha, y^H) = (1+\alpha, y^-)^H$, and $(1, y)(\alpha) = (1\alpha, y^-)$. The verification of the claim is now easily done.

(i) $(1, y{+}1)^H[\alpha] = H((1, y{+}1)[\alpha]) \rightsquigarrow^* H((1, y{+}1)[\alpha]{+}1) = H((1, y{+}1)(\alpha))$.

(ii) If $y = (\gamma, v)$ or $y = (\beta{+}1, z^-)$, then $(1, y)^H[\alpha] = (1, y^H{+}1)[\alpha] = (1{+}\alpha, y^H) = (1{+}\alpha, y^-)^H = H((1, y)(\alpha))$.

□

III.5 Properties of functionals $(H_x : x \in Q^H)$

In this section, we show that the properties of $(H_x : x \in Q_2^H)$ shown in Section II.3 extend to the general case. Lemma III.5.7 (as well as Definition III.5.5 and Lemma III.5.6) is only used when we build a notation system (cf. Chapter IV). Again, it is assumed that $f \in \Omega^{(0)}$, and for $x \in Q^H$, $f_x = H_x(f)$. Also recall that we identify a normal function with its range and that $f' = \{\alpha : f(\alpha) = \alpha\}$.

Lemma III.5.1. *If $y \rightsquigarrow_r^* x \in Q^H$, then we have for each n, $H_x^{+n} \subseteq H_y^{+n}$.*

Proof By induction along $\rightsquigarrow^*$. If $x = y$, the claim holds trivially, hence assume that $y \rightsquigarrow^* x$. We do a case distinction on the form of x.

(i) $y \rightsquigarrow^* x =_{NF} (1, x') \circ z$. If $x' = q_0$, then $y \rightsquigarrow_r^* z$ and by I.H. $H_z^{+n} \subseteq H_y^{+n}$. Thus, $H_x^{+n} = \mathsf{It}_{n+1} \circ H_z^{+n} \subseteq H_z \subseteq H_y$. If $x' = q_1$, then $y \rightsquigarrow_r^* (1, q_0) * z = x[0]$. Using the I.H. and that $\mathsf{It}_{n+2}(\mathsf{It}_{n+1}) \subseteq \mathsf{It}_{n+1}$, $H_x^{+n} \subseteq H_{x[0]}^{+n} \subseteq H_y^{+n}$. If $x' \neq q_0$ and $x' \neq q_1$, then there is an $\alpha < o(x)$, so that $y \rightsquigarrow_r^* x[\alpha] = (1, x'[\alpha]) * z$. By (i), $H_{x'}^{+(n+1)} \subseteq H_{x'[\alpha]}^{+(n+1)}$, and so $H_x^{+n} = H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}) \circ H_z^{+n} \subseteq H_{x'[\alpha]}^{+(n+1)}(\mathsf{It}_{n+1}) \circ H_z^{+n} = H_{x[\alpha]}^{+n} \subseteq_{IH} H_y^{+n}$.

(ii) $y \rightsquigarrow^* x =_{NF} (\gamma, x') * z$. There is an $\alpha < o(x)$, so that $y \rightsquigarrow_r^* x[\alpha] = (1{+}\alpha, x') \circ z$. As $H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}) \in \Omega^{(1)}$, $(H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}))^\gamma \subseteq (H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}))^{1+\alpha}$, and so $H_x^{+n} = (H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}))^\gamma \circ H_z^{+n} \subseteq (H_{x'}^{+(n+1)}(\mathsf{It}_{n+1}))^{1+\alpha} \circ H_z^{+n} \subseteq_{IH} H_y^{+n}$.

□

Lemma III.5.2. *If $x \in Q^H$, $\deg(x) > 1$ and $o(x) = 1$, then $f_x(\alpha) = f_{x[1+\alpha]}(0)$.*

Proof Assume that $\deg(x) = m{+}2$. As $x \in Q^H$ and $o(x) = 1$, we have that $x =_{NF} L((1, q_0) \circ_1 (1, x_1) \circ_1 \ldots \circ_1 (1, x_{m+1})) \circ x_{m+2}$. Using Corollary III.1.11, we see that $x = (1, x_1{+}1) \circ z$ for $z := L((1, x_2) \circ_1 \ldots \circ_1 (1, x_{m+1})) \circ x_{m+2}$, so $x[1{+}\alpha] = (2{+}\alpha, x_1) \circ z = ((2{+}\alpha, x_1) \circ_1 \ldots \circ_1 (1, x_{m+1})) \circ x_{m+2}$. By Lemma III.2.9,

$$\begin{aligned}
f_x(\alpha) &= ((\mathsf{It}_{m+2}, H_{(1,x_1)}^{+m}, H_{(1,x_2)}^{+(m-1)}, \ldots, H_{(1,x_{m+1})}^{+0}) \circ H_{x_{m+2}})[f, \alpha] \\
&= (((H_{(1,x_1)}^{+m})^{2+\alpha}, H_{(1,x_2)}^{+(m-1)}, \ldots, H_{(1,x_{m+1})}^{+0}) \circ H_{x_{m+2}})[f, 0] \\
&= ((H_{(2+\alpha,x_1)}^{+m}, H_{(1,x_2)}^{+(m-1)}, \ldots, H_{(1,x_{m+1})}^{+0}) \circ H_{x_{m+2}})[f, 0] = f_{x[1+\alpha]}(0).
\end{aligned}$$

□

Lemma III.5.3. *For each $x \in Q^H$ with $o(x) = \gamma$, we have*

(i) if $\xi < \gamma$, then $f_{x[\xi+1]} \subseteq f_{x[\xi]+1}$,

(ii) if $\xi < \gamma$, then $f_{x[\xi+2]} \subseteq f'_{x[\xi]}$,

(iii) $f_x = \bigcap_{\xi<\gamma} f_{x[\xi]} = \bigcap_{\xi<\gamma} f'_{x[\xi]}$.

Proof (i) $x = L((\gamma, x_0) \circ_1 \ldots \circ_1 x_m) \circ z$. So $x[\xi + 1] = L(q_1 \circ (1+\xi, x_0) \circ_1 \ldots \circ_1 x_m) \circ z$. Thus, $H_x = \mathsf{It}_{m+2} \circ H_{(1+\xi,x_0)}[\ldots] \subseteq \mathsf{it} \circ H_{(1+\xi,x_0)}[\ldots] = H_{x[\xi]+1}$.

(ii) By (i), $f_{x[\xi+1]} \subseteq \mathsf{it}(f_{x[\xi]})$. Since sh is monotone (cf. Lemma I.3.17), $\mathsf{sh}(f_{x[\xi+1]}) \subseteq (\mathsf{sh} \circ \mathsf{it})(f_{x[\xi]}) = f'_{x[\xi]}$. And as $\mathsf{it} \subseteq \mathsf{sh}$ (cf. Lemma I.3.15), $f_{x[\xi+2]} \subseteq \mathsf{sh}(f_{x[\xi+1]})$. (iii) If $o(x) = \gamma$, then $x =_{NF} L((\gamma, y) \circ_1 (1, y_1) \circ_1 \ldots \circ_1 (1, y_m)) \circ x_{m+1}$. So the first equality follows by Lemma III.2.9 and the definition of iteration of functionals. The second follows using (ii) and that $f' \subseteq f$ (so $f_{x[\alpha+2]} \subseteq f'_{x[\alpha]} \subseteq f_{x[\alpha]}$). □

Lemma III.5.4. *For each $x \in Q^H$ and each $y \in Q^H$ with $o(y) = \delta_0+\gamma$, we have*

(i) $f_x(\gamma) = \mathsf{sup}\{f_x(\xi) : \xi < \gamma\}$,

(ii) $f_y(0) = \mathsf{sup}\{s_0(\xi) : \xi < \gamma\}$ and $f_y(\alpha+1) = \mathsf{sup}\{s_{f_y(\alpha)+1}(\xi) : \xi < \gamma\}$, where $s_\beta(0) = \beta+1$, $s_\beta(\xi+1) := f_{y[\delta_0+\xi]}(s_\beta(\xi))$ and $s_\beta(\gamma') := \mathsf{sup}_{\xi<\gamma'} s_\beta(\xi)$.

Proof Completely analogous to the proof of Lemma II.3.5. □

The ordinals that occur in the name x are called the components of x, and are defined recursively as follows. Further, $|x|$ is the largest component of x.

Definition III.5.5. *$k(q_0) := \emptyset$, if $x = \langle(\alpha, y)\rangle$, then $k(x) := \{\alpha\} \cup k(y)$, and $k(\langle x_1, \ldots, x_n\rangle) = k(x_1) \cup \ldots \cup k(x_n)$. And $|x| := \mathsf{max}(k(x))$, where $\mathsf{max}(\emptyset) := -1$ (where -1 is below every ordinal, and $(-1)+1 := 0$).*

The following is readily observed.

Lemma III.5.6. *For all $x \in Q^H$, $0 < \alpha$, and each $f \in \Omega^{(0)}$, $f_x(\alpha) > f_x(0) \geq |x|$.*

For later reference, we also note the following.

Lemma III.5.7. *If $x \leq^H y$ and $|x| < \gamma \in f'_y$, then $\gamma \in f'_x$.*

Proof Fix x and assume that $x \leq^H y$ and $|x| < \gamma \in f'_y$. We show the claim by induction on y w.r.t. the ordering $(Q^H, <^H)$. If $x = y$, the claim holds trivially, and if $x <^H y$, we do a case distinction on the form of y. Thereby, we use that for $z \in Q^H$ and $\xi < \gamma$, $z[\xi] = (z[\gamma])[\xi]$.

(i) $y = z{+}1$. As $x <^H y = z{+}1$, we have $x \leq^H z$. Further, with $f_{z+1} = \mathsf{it}(f_z) \subseteq f_z$, also $f'_{z+1} \subseteq f'_z$ by Lemma I.3.17. Therefore, $\gamma \in f'_y \subseteq f'_z$. By I.H., the claim holds for z, thus $\gamma \in f'_x$.

(ii) $\deg(y) > 1$ and $o(y) = 1$. As $x <^H y$ and $|x| < \gamma$, there is a $\beta < \gamma$ so that $x <^H y[\beta]$ (cf. Lemma III.4.15). Since $\gamma = f_y(\gamma) = f_{y[\gamma]}(0) \in \bigcap_{\xi<\gamma} f_{y[\xi]}$, we obtain $\gamma \in f_{y[\beta+2]} \subseteq f'_{y[\beta]}$ by Lemma III.5.3 (ii). Applying the by I.H to $y[\beta{+}2]$ yields $\gamma \in f'_x$.

(iii) $\deg(y) = m{+}1$ and $o(y) = \gamma'$. Then $y = z[\gamma']$ for some z, and $\gamma \in f'_y \subseteq f_{z[\gamma']} = \bigcap_{\xi<\gamma'} f_{z[\xi]} = \bigcap_{\xi<\gamma'} f'_{z[\xi]}$ by Lemma III.5.3 (iii). Further, by Lemma III.4.15, there is a $\beta < \gamma'$ so that $x <^H z[\beta]$. Applying the I.H. to $f_{z[\beta]}$ yields $\gamma \in f'_x$.

□

"About equal"

The following interlude mentions some properties of the functionals H_x^{+m} that we only use for motivational purposes. For instance, we treat $1{+}q_2$ as q_2, although one could regard $1{+}q_2$ as a name of $\mathsf{It}(\mathsf{it}) \circ \mathsf{it}$ which is different from $\mathsf{It}(\mathsf{it})$, since $(\mathsf{It}(\mathsf{it}) \circ \mathsf{it})(f, n) = \mathsf{it}^{2+n+1}(f, 0) = (\mathsf{It}(\mathsf{it})(f, n{+}1)$. However, $\mathsf{It}(\mathsf{it}) \circ \mathsf{it}$ and $\mathsf{It}(\mathsf{it})$ are "about equal" in the following sense.

Definition III.5.8. *Let $m \in \mathbb{N}$. For each $f \in \Omega^{(0)}$, $(m{+}f)(\alpha) := f(m{+}\alpha)$, and if $[F, \vec{F}, f] \in \Omega^{\leq(n+1)}$, then $(m{+}F)[\vec{F}, f, \alpha] := F[\vec{F}, f, m{+}\alpha]$.*

Now, we say that $F, G \in \Omega^{(n+1)}$ are "about equal" if there is an $m \in \mathbb{N}$ so that $F \leq G \leq m{+}F$ or $G \leq F \leq n{+}G$.

Note that "about equal" is an equivalence relation.

Next, we show that if $\deg(x) > 1$, then H_x and $(H_x \circ \mathsf{it})$ are "about equal". We start with two auxiliary claims. Recall that $x[\alpha] \rightsquigarrow_r^* x$ and thus $f_{x[\alpha]} \leq f_x$.

Lemma III.5.9. *For each $x \in Q^H$, H_x^{+m} is $\leq$-monotone. That is if $F, G \in \Omega^{(m)}$ with $F \leq G$, then $H_x^{+m}(F) \leq H_x^{+m}(G)$.*

Proof An easy induction on the build up of H_x^{+m}. □

Lemma III.5.10. *For each $x \in Q^H$, $H_x \circ \mathsf{it} \leq H_{x+1}$.*

Proof By induction on $\rightsquigarrow^*$. The claim is trivial for $x = q_0$. If $x = y{+}1$, then $H_{y+1} \circ \mathsf{it} = (\mathsf{it} \circ H_y) \circ \mathsf{it} = \mathsf{it} \circ (H_y \circ \mathsf{it}) \leq_{IH} \mathsf{it} \circ H_{y+1}$. If $\deg(x) = 1$ and $o(x) = \gamma$, then $H_x \circ \mathsf{it} = \bigcap_{\alpha<\gamma} H_{x[\alpha]} \circ \mathsf{it} = \bigcap_{\alpha<\gamma} H_{x[\alpha]}$ as $H_{x[\alpha]} \supseteq H_{x[\alpha]} \circ \mathsf{it} \supseteq H_{x[\alpha]} \circ H_{x[\alpha]}$. And if $\deg(x) > 1$, then $(H_x \circ \mathsf{it})(f, \alpha) = (H_{x[1+\alpha]} \circ \mathsf{it})(f, 0) \leq_{IH} (\mathsf{it} \circ H_{x[1+\alpha]})(f, 0) \leq (\mathsf{it} \circ H_x)(f, \alpha)$. □

Lemma III.5.11. *If* $\deg(x) > 1$*, then* $H_x \circ \mathsf{it} \leq 1{+}H_x$*.*

Proof Using Lemmas III.5.10,III.5.9 and III.5.2, we readily see that $(H_x \circ \mathsf{it}, f, \alpha) = (H_{x[1+\alpha]} \circ \mathsf{it}, f, 0) \leq (\mathsf{it} \circ H_{x[1+\alpha]}, f, 0) = (H_{x[1+\alpha]+1}, f, 0) \leq (H_{[x[1+\alpha+1]}, f, 0) = (1{+}H_x, f, \alpha)$. □

Lemma III.5.12. *If* $(\omega, q_0) \leq x$*, then* H_x *and* $H_x \circ \mathsf{it}$ *are "about equal".*

Proof By the above lemma and since with $H_{(\gamma,x)} \circ \mathsf{it} = H_{(\gamma,x)}$, also $H' \circ H_{(\gamma,x)} \circ H_z \circ \mathsf{it} = H' \circ H_{(\gamma,x)} \circ H_z$. □

Lemma III.5.13. *Let* $z := (x \circ_1 (1, y)) \in Q^H$*. Then,* $H^{+m}_{z^H}$ *and* $H^{+(m+1)}_{x^*}(H^{+m}_{(1,y)^H})$ *are "about equal", where* $x^* := x{+}\mathsf{corr}(x)$ *(cf. Definition III.4.20).*

Proof Let $z = x \circ_1 (1, y) = (1, x \circ y)$. We consider the following cases. If $x = (n, q_0)$, then $\mathsf{corr}(x) = 0$ and $\mathsf{corr}(y) = \mathsf{corr}(x \circ y)$, so $H^{+m}_{z^H} = H^{+(m+1)}_{x^*}(H^{+m}_{(1,y)^H})$. And if $x = (\gamma, x') * z{+}n$ or if $\deg(x) > 1$, then $\mathsf{corr}(x) = \mathsf{corr}(x \circ y)$, so depending on $\mathsf{corr}(y)$, $H^{+m}_{z^H}$ is either $H^{+(m+1)}{}_{x^*}(H^{+m}_{(1,y)^H})$ or $(H^{+(m+1)}_{x^*} \circ \mathsf{It}_{m+1})(H^{+m}_{(1,y)^H})$. In both cases, the claim follows by Lemma III.5.12. □

Corollary III.5.14. *Assume that* $z = L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_m))$*. Then we have that* $H^{+n}_{z^H}$ *and* $(H^{(m+n+1)}_{x^*}, H^{(m+n)}_{(1,y_0)^H}, \ldots, H^{+n}_{(1,y_m)^H})$ *are "about equal".*

Proof By induction on m. For $m = 0$ the claim is by Lemma III.5.13. Now we assume that the claim holds for m and we prove it for $m{+}1$. Thereto, we let $z_m := L(x \circ_1 (1, y_0) \circ_1 \ldots, \circ_1 \ldots \circ_1 (1, y_m))$, so that $z = (z_m \circ_1 (1, y_{m+1}))$. By Lemma III.5.13, and since $\deg(z_m) > 1$ and so $z^*_m = z^H_m$, $H^{+n}_{z^H}$ and $H^{+(n+1)}_{z^H_m}(H^{+n}_{(1,y)^H})$ are "about equal". By I.H.,

$$H^{+(n+1)}_{z^H_m} \text{ and } (H^{(m+n+2)}_{x^*}, H^{(m+n+1)}_{(1,y_0)^H}, \ldots, H^{+(n+1)}_{(1,y_m)^H})$$

are "about equal". Applying both sides to $H^{+n}_{(1,y_{m+1})^H}$ yields the claim. □

The point of this corollary is the following. For $x \in Q^*_1$, $\mathsf{Prv}_2(x)$ states that

$$\mathsf{Op}_{(1,y)} \text{ proves } H_{(1,y)^H} \quad \Longrightarrow \quad \mathsf{Op}_x \circ \mathsf{Op}_{(1,y)} \text{ proves } (H_{x^*}, H_{(1,y)^H}).$$

We will lift this definition with the idea that "$\mathsf{Op}^{+(m+1)}_x$ proves $H^{+(m+1)}_x$" expresses that if $z = L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_m))$ and for each $i \leq m$, "$\mathsf{Op}^{+(m-i)}_{(1,y_i)}$ proves $H^{+(m-i)}_{(1,y_i)}$", then "$\mathsf{Op}^{+(m+1)}_x \circ \mathsf{Op}^{+m}_{(1,y_0)} \ldots \circ \mathsf{Op}^{+0}_{(1,y_m)}$ proves $(H^{+(m+1)}_x, \ldots, H^{+0}_{(1,y_m)})$". Since $\mathsf{Op}^{+(m+1)}_x \circ \mathsf{Op}^{+m}_{(1,y_0)} \ldots \circ \mathsf{Op}^{+0}_{(1,y_m)}$ iff Op_z, and we also have that Op_z proves H_{z^H}, H_{z^H} and $(H^{+(m+1)}_x, \ldots, H^{+0}_{(1,y_m)})$ should be "about equal".

III.6 The operations $(\mathsf{Op}_x : x \in Q^*_{N_0})$

As in the case $N_0 = 2$, it is henceforth assumed that we have primitive recursive relations that are formalized versions of $(Q, <)$, $\rightsquigarrow$, $\rightsquigarrow^*$, and primitive recursive functions formalizing $\deg(x)$, $\circ$ and $\cdot^H$. To emphasis that we now work within a formal theory, we write $\alpha \lhd \beta$ for $\alpha < \beta$. The other function- and relation symbols are overloaded.

III.6.1 The proper definition of $(\mathsf{Op}_x : x \in Q^*_{N_0})$

We extend the definition of the $\mathsf{L}_2(P)$-formula $\varphi(u)$ given in Definition II.4.1 in an obvious way, and show that the resulting formula indeed represents the operations Op_x. However, since there is no $\mathsf{L}_2(\mathsf{P})$-formula $\vartheta(u)$ so that for all n, $\mathsf{Op}^{\vartheta}_{n+1} \Leftrightarrow \mathsf{p}_{n+1}$, all the following is relative to some fixed $N_0 \in \mathbb{N}$, as we use the $\mathsf{L}_2(\mathsf{P})$-formula

$$\vartheta^{N_0}(u) = u \leq N_0 \wedge \bigwedge_{n<N_0} (u = n{+}1 \rightarrow \varphi_{\mathsf{p}_{n+1}})$$

to represent for each $1 \leq n \leq N_0$, the basic operation p_n by $\varphi^{N_0}(\overline{n})$, where $\varphi_{\mathsf{p}_{n+1}}$ is as fixed in Definition I.2.15. Consequently, we can only represent operations with names from $Q^*_{N_0}$.

Now, we supplement the proper definition of Op^{+n}_x $(x \in Q^*_{N_0-n})$, which supersedes the provisional Definition II.1.12. Again, $\varphi^{f^+,\rightsquigarrow^*\restriction Q_1,\rightsquigarrow\restriction Q_1,\vartheta}(u)$ is the formula defined in the Appendix (Theorem A.1.2 and Definition A.1.11).

Definition III.6.1. *For each $n < N_0$, we let $f^{+n}(y, x) := \deg(x){+}n$, and*

$$\varphi^{+n}(u) := \varphi^{f^{+n},\rightsquigarrow^*\restriction Q_{N_0-n},\rightsquigarrow\restriction Q_{N_0-n},\vartheta^{N_0}}(u).$$

Then, $\mathsf{Op}^{+n}_x(\breve{\mathsf{T}}) := \mathsf{Op}^{\varphi^{+n}}_x(\breve{\mathsf{T}})$, $\mathsf{Op}_x(\breve{\mathsf{T}}) := \mathsf{Op}^{\varphi^{+0}}_x(\breve{\mathsf{T}})$ and $\mathsf{Op}^+_x(\breve{\mathsf{T}}) := \mathsf{Op}^{\varphi^{+1}}_x(\breve{\mathsf{T}})$. Further, we define $\breve{\mathsf{T}}^{+n}_x := (x = q_0 \wedge \breve{\mathsf{T}}) \vee (x \neq q_0 \wedge \mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$.

The following is again essentially an instance of the Representation Theorem. The addition of $n < N_0$ in (i) of the lemma below is by the definition of $\vartheta^{N_0}(u)$, as $0 \rightsquigarrow^* x \wedge N_0 \leq n \wedge \vartheta^{N_0}_{\breve{\mathsf{T}}\restriction U}(\deg(x){+}n) \leftrightarrow \bot$. Also note that if $x \in Q_n$, then $\mathsf{Wo}_{\rightsquigarrow^*}(x)$ iff $\mathsf{Wo}_{\rightsquigarrow^*\restriction Q_n}(x)$.

Lemma III.6.2. *The map $\breve{\mathsf{T}} \mapsto \mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$ is an operation that satisfies the following properties (provable in T^ϵ).*

(i) $\mathsf{Op}^{+n}_x(\breve{\mathsf{T}}) \rightarrow n < N_0 \wedge 0 \rightsquigarrow^* x \wedge x \in Q^*_{N_0-n} \wedge \mathsf{Wo}_{\rightsquigarrow^*}(x)$,

(ii) $\mathsf{Op}^{+n}_{q_1} \Leftrightarrow \mathsf{p}_{n+1}$,

(iii) if $q_1 \rightsquigarrow^* x$ *and* $\deg(x) = m{+}1$, *then* $\mathsf{Op}_x^{+n} \Leftrightarrow (\forall\alpha \lhd o(x))(\mathsf{p}_{m+n+1} \circ \mathsf{Op}_{x[\alpha]}^{+n})$.

From a technical point of view, we only need the operations Op_x^{+n} for $x \in Q^*_{N_0-n}$. However, in order to compare the proper with the provisional definition, we also supplement a proper definition of $\mathsf{Op}_{y^-}^{+(m,n)}$ for each $y^- \in P_{N_0-n}$.

Definition III.6.3. *For each* $0 \leq m < n$ *and* $y^- \in P_{N_0-n}$,

$$\mathsf{Op}_{y^-}^{+(m,n)} := (\forall\alpha \lhd \gamma)\mathsf{p}_{m+1}\mathsf{Op}_{y[\alpha]}^{+n}.$$

Since $\mathsf{p}_{m+1}(\breve{\mathsf{T}})$ is Π^1_{m+n+2}, the following is readily observed.

Lemma III.6.4. *Let* $y^- \in P$ *and* $x \in Q^*_{N_0}$ *with* $\deg(x) = m{+}1$. *Then,*

(i) for all $0 \leq k < n$, $\mathsf{Op}_{y^-}^{+(k,n)}(\breve{\mathsf{T}})$ *is* Π^1_{k+2},

(ii) for all n, $\mathsf{Op}_x^{+n}(\breve{\mathsf{T}})$ *is* Π^1_{m+n+2}.

In the sequel, we drop the subscript N_0. It is assumed to be big enough.

III.6.2 Properties of $(\mathsf{Op}_x : x \in Q^*)$

The next couple of lemmas are all shown using Theorem I.4.2. For all these proofs, we let $A(x)$ express the claim, and we proceed exactly as described in Subsection II.4.2 (cf. page 55).

Lemma III.6.5. *The following is provable in* T^ϵ. *For all* $x, y \in Q$,

(i) if $y \rightsquigarrow^* x$, *then* $\breve{\mathsf{T}}_x^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_y^{+n}$, *in particular, if* $x \in Q^*$, *then* $\mathsf{Op}_x^{+n} \Rightarrow \mathsf{p}_{n+1}$.

(ii) if $x \circ y \in Q^*$, *then* $\mathsf{T}_{x\circ y}^{+n} \leftrightarrow \mathsf{Op}_x^{+n}(\breve{\mathsf{T}}_y^{+n})$.

(iii) if $x \in Q^*$, $\mathsf{Op}_{(1,x)}^n \Leftrightarrow \mathsf{Op}_x^{n+1}\mathsf{p}_{n+1}$.

Proof (i) Let $A(x) := y \rightsquigarrow^* x \in Q^* \wedge \breve{\mathsf{T}}_x^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_y^{+n}$. If $x = y{+}1$, then $x[0] = y$, and the claim is by definition of $\breve{\mathsf{T}}_x^{+n}$. Otherwise, there is an $\alpha \lhd o(x)$ so that $y = x[\alpha]$ or $y \rightsquigarrow^* x[\alpha]$. In the first case, $\breve{\mathsf{T}}_x^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_y^{+n}$ by definition of $\breve{\mathsf{T}}_x^{+n}$ and since, in any case, $\mathsf{p}_{\deg(x)+n} \Rightarrow \mathsf{p}_{n+1}$. And if $x[\alpha] \rightsquigarrow^* x$, then the I.H. yields for each X, $\breve{\mathsf{T}}_{x[\alpha]}^{+n}\restriction X \rightarrow \mathsf{p}_1\breve{\mathsf{T}}_y^{+n}\restriction X$. Since p_{n+1} is an operation, we obtain $\mathsf{p}_{n+1}\breve{\mathsf{T}}_{x[\alpha]}^{+n} \rightarrow \mathsf{p}_{n+1}^2\breve{\mathsf{T}}_y^{+n}$. As $\breve{\mathsf{T}}_x^{+n}$ iff $(\forall\xi \lhd o(x))\mathsf{p}_{\deg(x)+n}\breve{\mathsf{T}}_{x[\xi]}^{+n}$ and $\mathsf{p}_{\deg(x)+n} \Rightarrow \mathsf{p}_{n+1}$, we have also $\breve{\mathsf{T}}_x^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_{x[\alpha]}^{+n}$. By Lemma I.2.12, $\mathsf{p}_{n+1}^2\breve{\mathsf{T}}_y^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_y^{+n}$, hence $\breve{\mathsf{T}}_x^{+n} \rightarrow \mathsf{p}_{n+1}\breve{\mathsf{T}}_y^{+n}$ follows.

(ii) Let $A(x) := q_0 \neq x \wedge x \circ y \in Q^* \rightarrow [\breve{\mathsf{T}}_{x\circ y}^{+n} \leftrightarrow \mathsf{Op}_x^{+n}(\breve{\mathsf{T}}_y^{+n})]$. If x or y is q_0, then the claim is trivial, so assume otherwise. Next, we let m so that $m{+}1 =$

$\deg(x) = \deg(x \circ y)$, and δ_0 so that $o(x \circ y) = \delta_0 + o(x)$ and thus by Lemma III.4.16, $(\forall \alpha \lhd o(x))((x \circ y)[\alpha] = x[\alpha] \circ y)$. Then, $\breve{\mathsf{T}}^{+n}_{x \circ y} \leftrightarrow (\forall \alpha \lhd o(x \circ y))\mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+n}_{(x \circ y)[\alpha]})$. Hence, $\breve{\mathsf{T}}^{+n}_{x \circ y} \to (\forall \alpha \lhd o(x))\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{x[\alpha] \circ y})$. For each $\alpha \lhd o(x)$ and each X, the I.H. yields $\mathsf{Op}^{+n}_{x[\alpha] \circ y}(\breve{\mathsf{T}}^{+n}) \restriction X \leftrightarrow \mathsf{Op}^{+n}_{x[\alpha]}(\breve{\mathsf{T}}^{+n}_y) \restriction X$. Since p_{m+1} is an operation, we obtain $(\mathsf{p}_{m+1} \circ \mathsf{Op}^{+n}_{x[\alpha] \circ y})(\breve{\mathsf{T}}^{+n})$ iff $(\mathsf{p}_{m+1} \circ \mathsf{Op}^{+n}_{x[\alpha]})(\breve{\mathsf{T}}^{+n}_y)$. Now $\breve{\mathsf{T}}^{+n}_{x \circ y} \to \mathsf{Op}^{+n}_x(\breve{\mathsf{T}}^{+n}_y)$ readily follows. For the converse direction, observe that $(\forall \xi \lhd \gamma)\mathsf{p}_{m+1}\mathsf{Op}^{+n}_{x[\alpha]}(\breve{\mathsf{T}}^{+n}_y)$ yields $(\forall \alpha \lhd \gamma)\mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+n}_{(x \circ y)[\delta_0 + \alpha]})$. Using (ii) yields $(\forall \xi \lhd \delta_0 + \gamma)\mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+n}_{(x \circ y)[\delta_0 + \alpha]})$.

(iii) Similar, using that $\deg(1, x) = \deg(x) + 1$ and $o(1, x) = o(x)$. $\square$

The next Lemma generalizes Lemma I.2.14.

Lemma III.6.6. *For each open Π^1_{n+2}-sentence $\breve{\mathsf{T}}'$,*

$$\mathsf{T}^\epsilon \vdash x \in Q^* \wedge \mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}}').$$

Proof Let $\breve{\mathsf{T}}'$ be Π^1_{n+2}, and $A(x) := x \in Q^* \wedge \mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}}')$. Trivially, we have $A(q_0)$, and if $x = q_1$, then by Lemma I.2.14, $\mathsf{p}_{n+2}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ yields $\mathsf{p}_{n+2}(\breve{\mathsf{T}}')$. Next, let $q_1 \neq x \in Q^*$ with $\deg(x) = m+1$, and assume $\mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ and $(\forall y \rightsquigarrow^* x)a(y)$. By definition, $\mathsf{Op}^{+(n+1)}_x(\breve{\mathsf{T}})$ iff $(\forall \alpha \lhd o(x))\mathsf{p}_{m+n+2}\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}})$. Hence for each $\alpha \lhd o(x)$, $\mathsf{p}_{m+n+2}(\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}})) \wedge \breve{\mathsf{T}}'$, thus $\mathsf{p}_{m+n+2}(\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}')$, again by Lemma I.2.14. Further, for $\alpha \lhd o(x)$, $x[\alpha] \rightsquigarrow^* x$, thus $a(x[\alpha])$, that is,

$$\forall X[(\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}') \restriction X \to \mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}}') \restriction X].$$

We obtain $\mathsf{p}_{m+n+2}(\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}') \to \mathsf{p}_{m+n+2}(\mathsf{Op}^{+(n+1)}_{x[\alpha]}(\breve{\mathsf{T}}'))$ for each $\alpha \lhd o(x)$, since p_{m+n+2} is an operation. Hence, $\mathsf{Op}^{+n}_x(\breve{\mathsf{T}}')$. Thus, we have $A(x)$. $\square$

Lemma III.6.7. *Let $(1, y) \in Q$ be a simple name of degree $m+2$. Then,*

(i) $\mathsf{Op}^{+(n+1)}_y \Rightarrow \mathsf{Op}^{+n}_{(1,y)}$.

(ii) $\mathsf{Op}^{+(n+1)}_{(1,y)} \Leftrightarrow \mathsf{Op}^{+n}_{(1,y)}\mathsf{p}_{n+1}$.

Proof (i) $\mathsf{Op}^{+(n+1)}_y(\breve{\mathsf{T}})$ implies $\mathsf{p}_{n+1}(\breve{\mathsf{T}})$ which is Π^1_{n+2}, thus by the above lemma, $\mathsf{Op}^{+(n+1)}_y(\breve{\mathsf{T}})$ implies $\mathsf{Op}^{+n}_y\mathsf{p}_{n+1}(\breve{\mathsf{T}})$, or in other words, $\mathsf{Op}^{+(n+1)}_y \Rightarrow \mathsf{Op}^{+n}_{(1,y)}$. (ii) $\mathsf{Op}^{+(n+1)}_{(1,y)} \Leftrightarrow \mathsf{Op}^{+n}_y\mathsf{p}_{n+1}$, so (i) implies $\mathsf{Op}^{+n}_{(1,y)}\mathsf{p}_{n+1}$. For the converse direction, note that $\mathsf{Op}^{+n}_{(1,y)}\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}^{+(n+1)}_y\mathsf{p}^2_{n+1}$. As $\mathsf{p}_{n+2} \Rightarrow \mathsf{p}_{n+1}$ by Lemma I.2.12, we further obtain that $\mathsf{Op}^{+(n+1)}_y\mathsf{p}^2_{n+1} \Rightarrow \mathsf{Op}^{+(n+1)}_y\mathsf{p}_{n+1}$, and the claim is by (i). $\square$

Corollary III.6.8. *For each open Π^1_{n+2}-sentence $\breve{\mathsf{T}}'$,*

$$\mathsf{T}^\epsilon \vdash y \in Q^* \wedge \mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}' \to \mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}').$$

Proof By Lemma III.6.5 (iii), $\mathsf{Op}^{+n}_{(1,y)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{y}\mathsf{p}_{n+1}$. Hence, $\mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}) \wedge \breve{\mathsf{T}}'$ yields $\mathsf{Op}^{+(n+1)}_{y}(\mathsf{p}_{n+1}(\breve{\mathsf{T}})) \wedge \breve{\mathsf{T}}'$, which by Lemma III.6.6 implies $\mathsf{Op}^{+(n+1)}_{y}(\breve{\mathsf{T}}')$, which in turn yields $\mathsf{Op}^{+n}_{(1,y)}(\breve{\mathsf{T}}')$ by Lemma III.6.7. □

The proofs of the next few results for a change, do not make (direct) use of Theorem I.4.2.

Lemma III.6.9. *The following is provable in T^ϵ.*

(i) *if $x^- \in P$, then $\mathsf{Op}_{(1,x)} \Rightarrow \mathsf{Op}_{(1,x^-)}$.*

(ii) $\mathsf{Op}^{+n}_{x\circ_1(1,y)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x} \circ \mathsf{Op}^{+n}_{(1,y)}$.

Proof (i) Assume that $\deg(x) = m+1$. Since $x^- \in P$, we have $o(x) = \gamma$, in particular, x is not of the form $y+1$, and therefore, $(1,x)[\alpha] = (1,x^-)[\alpha]$. Hence $\mathsf{Op}_{(1,x)} \Leftrightarrow (\forall\alpha \lhd o(x))\mathsf{p}_{m+2}\mathsf{Op}_{(1,x)[\alpha]} \Rightarrow (\forall\alpha \lhd o(x))\mathsf{p}_{m+1}\mathsf{Op}_{(1,x^-)[\alpha]} \Leftrightarrow \mathsf{Op}_{(1,x^-)}$. (ii) If $y = q_0$, then $\mathsf{Op}^{+n}_{(x\circ_1(1,q_0))} \Leftrightarrow \mathsf{Op}^{+n}_{(1,x)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x}\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x} \circ \mathsf{Op}^{+n}_{(1,q_0)}$, and if $q_0 \rightsquigarrow^* y$, then we have $\mathsf{Op}^{+n}_{(1,x\circ y)} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x\circ y}\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x}\circ\mathsf{Op}^{+(n+1)}_{y}\mathsf{p}_1 \Leftrightarrow \mathsf{Op}^{+(n+1)}_{x}\circ\mathsf{Op}^{+n}_{(1,y)}$. □

Below, we present internal variants of Lemma III.3.9 (ii) and Corollary III.3.11. With the above properties of Op_x at hand, the proofs are almost literally the same.

Lemma III.6.10. *If $q_0 \neq x$ and $x \circ_m y \in Q$, then $\mathsf{Op}^{+n}_{x\circ_m y} \Leftrightarrow \mathsf{Op}^{+(m+n)}_{x} \circ \mathsf{Op}^{+n}_{y}$ if $y \neq q_0$, and $\mathsf{Op}^{+n}_{x\circ_m y} \Leftrightarrow \mathsf{Op}^{+(m+n)}_{x}$, otherwise.*

Lemma III.6.11. T^ϵ *proves: if $x = L(y_0 \circ_1 \ldots \circ_1 y_m) \circ z \in Q^*$, then,*

$$\mathsf{Op}^{+n}_{x} \Leftrightarrow (\mathsf{Op}^{+(n+m)}_{y_0} \circ \mathsf{Op}^{+(n+m-1)}_{y_1} \circ \ldots \circ \mathsf{Op}^{+n}_{y_m}) \circ \mathsf{Op}^{+n}_{z}.$$

Our next goal is to lift Lemma II.4.9 to the general case.

Lemma III.6.12. *Assume that x is a simple name of degree two. Then,*

(i) $\mathsf{Op}^{+n}_{x} \Rightarrow \mathsf{Op}^{+n}_{x(0)}$,

(ii) $\mathsf{Op}^{+n}_{x(0)} \circ \mathsf{Op}^{+n}_{x(\alpha)} \Rightarrow \mathsf{Op}^{+n}_{x(\alpha+1)}$,

(iii) *if $o(x) = 1$, then $\mathsf{Op}^{+n}_{x[\gamma]} \Leftrightarrow (\forall\alpha \lhd \gamma)\mathsf{Op}^{+n}_{x(\alpha)}$.*

(iv) *if $o(x) \in \mathsf{Lim}(\Omega)$, then $\mathsf{Op}^{+n}_{x(\gamma)} \Leftrightarrow (\forall\alpha \lhd \gamma)\mathsf{Op}^{+n}_{x(\alpha)}$.*

Proof Almost literally as the corresponding proof of Lemma II.4.8, now using that $(\forall\alpha\lhd o(x))\mathsf{Op}_x^{+n}$ iff $\mathsf{p}_{n+2}\mathsf{Op}_{x[\alpha]}^{+n}$ (cf. Lemma III.6.2 (iii)), and Lemmas III.6.5, III.4.13 instead of II.4.4, II.2.14. $\square$

Recall that if $\deg(x) > 1$ and $\beta \lhd \alpha$, then by Lemma III.4.19 (ix), $x(\beta) \rightsquigarrow^* x(\alpha)$, and so by Lemma III.6.5 (i), $\mathsf{Op}_{x(\alpha)}^{+n} \Rightarrow \mathsf{Op}_{x(\beta)}^{+n}$.

Lemma III.6.13. $\mathsf{T}^\epsilon \vdash \deg(x) > 1 \wedge \check{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\{\alpha : \check{\mathsf{T}}_{x(\alpha)}\})$.

Proof This time, assume that $\deg(x) = m+2$, and $x =_{NF} (1,y) \circ_m z$, so Op_x iff $\mathsf{Op}_{(1,y)}^{+m} \circ \mathsf{Op}_z$. Now the claim is shown completely analogously to Lemma II.4.9 using the above Lemma, and Corollary III.6.8 and Lemma III.4.13 instead of Corollary II.4.7 and Lemma II.2.14. $\square$

III.6.3 The proper and provisional Definition of Op_x agree

This subsection is devoted to the proof that the proper Definition III.6.1 agrees with the provisional Definition III.3.4. Again, from a technical point of view, none of these results are used in the sequel.

Lemma III.6.14. *The following is provable in* T^ϵ.

(i) For all $x \in Q^*$, $\mathsf{Op}_x^{+n}\mathsf{p}_{n+1} \Rightarrow \mathsf{Op}_x^{+n}$.

(ii) If $q_2 \neq x \in Q^*$, *then* $\mathsf{Op}_x^{+n} \Rightarrow \mathsf{Op}_x^{+n}\mathsf{p}_{n+1}$.

This leads for instance to the following results which correspond to Lemmas II.4.11, II.4.12, and Corollaries II.4.13 and II.4.14.

Lemma III.6.15. T^ϵ *proves: for all* $x \in Q^*$,

(i) $\mathsf{Op}_x^{+n} \circ \mathsf{Op}_x^{+n} \Rightarrow \mathsf{Op}_x^{+n}$,

(ii) if $0 \lhd \beta \lhd \alpha$, *then* $(\mathsf{Op}_x^{+n})^\alpha \Rightarrow (\mathsf{Op}_x^{+n})^\beta$, *and* $(\mathsf{Op}_x^{+n})^\gamma \Leftrightarrow (\forall\xi \lhd \gamma)(\mathsf{Op}_x^{+n})^{1+\xi}$.

The following results corresponds to Lemma II.4.15 and II.4.16. The proofs are omitted as they are easily lifted from the case $n = 2$ to the general case.

Lemma III.6.16. T^ϵ *proves: if* $q_1 \neq x \in Q^*$ *and* $0 \lhd \alpha$, *then*

(i) $\mathsf{p}_{n+1}(\mathsf{Op}_x^{+n})^\alpha \Rightarrow (\mathsf{p}_{n+1}\mathsf{Op}_x^{+n})^\alpha$,

(ii) $(\mathsf{p}_{n+1}\mathsf{Op}_x^{+n})^{\alpha+1} \Rightarrow \mathsf{p}_{n+1}(\mathsf{Op}_x^{+n})^\alpha$.

Lemma III.6.17. T^ϵ *proves: if* $x \in Q^*$, *then* $\mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1}^2$.

Lemma III.6.18. T^ϵ *proves: for all* $x \in Q^*$, $\mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1} \Leftrightarrow \mathsf{Op}_{(1,x)}^{+n}$.

Now we prove the aforementioned equivalence the proper and the provisional definition of Op_x^{+n}. As in the case $n = 2$, most work is done in the proof of the next two lemmas.

Lemma III.6.19. *For each* $n \in \mathbb{N}$, T^ϵ *proves the following: if* $x \in Q^*$, *then*

$$\mathsf{Op}_{(1,x)}^{+n} \circ (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma \Leftrightarrow (\mathsf{Op}_{(1,x)}^{+n})^\gamma.$$

Proof Recall that $(\mathsf{Op}_x^{+n})^\gamma \Leftrightarrow (\forall \xi \lhd \gamma)(\mathsf{Op}_x^{+n})^{1+\xi}$ (cf. Lemma III.6.15 (ii)). To show the $\Rightarrow$-direction, fix an $\eta \rhd 0$. Since $\mathsf{Op}_{(1,x)}^{+n}$ iff $\mathsf{Op}_{(1,x)}^{+n}\mathsf{p}_{n+1}$ (cf. Lemma III.6.14), and $(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma \Rightarrow (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^{\eta+1} \Rightarrow_{L.III.6.16} \mathsf{p}_{n+1}(\mathsf{Op}_{(1,x)}^{+n})^\eta$, we have that $\mathsf{Op}_{(1,x)}^{+n} \circ (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma \Rightarrow (\mathsf{Op}_{(1,x)}^{+n})^{\eta+1} \Rightarrow_{L.III.6.15} (\mathsf{Op}_{(1,x)}^{+n})^\eta$. For the converse direction observe that $(\mathsf{Op}_{(1,x)}^{+n})^\gamma \Rightarrow (\forall \xi \lhd \gamma)(\mathsf{Op}_{(1,x)}^{+n})^{\xi+1} \Rightarrow (\forall \xi \lhd \gamma)\mathsf{p}_{n+1}(\mathsf{Op}_{(1,x)}^{+n})^\xi \Rightarrow (\forall \xi \lhd \gamma)(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\xi$. Let $C := (\forall \xi \lhd \gamma)(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\xi(\breve{\mathsf{T}})$, which is Π^1_{n+2}, and note that $(\mathsf{Op}_{(1,x)}^{+n})^\gamma(\breve{\mathsf{T}})$ implies $\mathsf{Op}_x^{+(n+1)}(\mathsf{p}_{n+1}(\breve{\mathsf{T}})) \wedge C$. Hence Lemma III.6.6 further yields $\mathsf{Op}_x^{+(n+1)}(C)$, therefore also $\mathsf{Op}_x^{+(n+1)}\mathsf{p}_{n+1}(C)$, that is, $\mathsf{Op}_{(1,x)}^{+n} \circ (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma(\breve{\mathsf{T}})$. $\square$

Lemma III.6.20. *For each* $n \in \mathbb{N}$, T^ϵ *proves the following: if* $x \in Q^*$, *then*

(i) $\mathsf{Op}_{(m+1,x)}^{+n} \Leftrightarrow (\mathsf{Op}_{(1,x)}^{+n})^{m+1}$, *and* $\mathsf{Op}_{(\gamma+m+1,x)}^{+n} \Leftrightarrow (\mathsf{Op}_{(1,x)}^{+n})^{\gamma+m}$.

(ii) $\mathsf{Op}_{(\gamma,x)}^{+n} \Leftrightarrow (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma$.

Proof Using Theorem I.4.2. We let $A(\alpha)$ so that $A(m{+}1)$ and $A(\gamma{+}m{+}1)$ express (i), and $A(\gamma)$ expresses (ii). First, we show $A(\gamma{+}1)$, i.e., $\mathsf{Op}_{(\gamma,x)}^{+n} \Leftrightarrow (\mathsf{Op}_{(1,x)}^{+n})^{\gamma+1}$. By I.H. we have $\mathsf{Op}_{(\gamma,x)}^{+n}(\breve{\mathsf{T}})\restriction X \Leftrightarrow (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma(\breve{\mathsf{T}})\restriction X$. As $\mathsf{Op}_{(1,x)}^{+n}$ is an operation, $\mathsf{Op}_{(1,x)}^{+n} \circ \mathsf{Op}_{(\gamma,x)}^{+n} \Leftrightarrow \mathsf{Op}_{(1,x)}^{+n} \circ (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma$ follows. By Lemma III.6.5 (ii) and Lemma III.6.19, we obtain $\mathsf{Op}_{(\gamma+1,x)}^{+n} \Leftrightarrow (\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\gamma$. $A(1)$ holds trivially, and $A(m{+}1)$ and $A(\gamma{+}m{+}2)$ are directly from the I.H.

Now we show $A(\gamma)$. Using the definition of $\mathsf{Op}_{(\gamma,x)}^{+n}$, Lemma III.6.5 (iii) and III.6.15, this amounts to show that

$$(\forall \xi \lhd \gamma)\mathsf{p}_{n+1}\mathsf{Op}_{(1+\xi,x)}^{+n} \Leftrightarrow (\forall \xi \lhd \gamma)(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^{1+\xi}.$$

To show that $\Rightarrow$-direction, fix a $\eta \lhd \gamma$ with $\eta \rhd 0$. $(\forall \xi \lhd \gamma)\mathsf{p}_{n+1}\mathsf{Op}_{(1+\xi,x)}^{+n}$ entails $\mathsf{p}_{n+1}\mathsf{Op}_{(\eta+1,x)}^{+n}$. Using the I.H. yields $\mathsf{p}_{n+1}(\mathsf{Op}_{(1,x)}^{+n})^\eta$, and $(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^\eta$ follows by Lemma III.6.16. For the converse direction, also fix an $\eta \lhd \gamma$ with $\eta \rhd 0$. Note that $(\forall \xi \lhd \gamma)(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^{1+\xi}$ entails $(\mathsf{p}_{n+1}\mathsf{Op}_{(1,x)}^{+n})^{\eta+1}$. Lemma III.6.16 yields $\mathsf{p}_{n+1}(\mathsf{Op}_{(1,x)}^{+n})^\eta$. Using the I.H. and possibly Lemma III.6.5 (ii) and Lemma III.6.15 yields $\mathsf{p}_{n+1}\mathsf{Op}_{(\eta,x)}^{+n}$. $\square$

Theorem III.6.21. *For each n, all $\langle x_1,\dots,x_k\rangle, x\in Q^*$ $(k>1)$, each $y^-\in P$, each $v\in Q^*\cup P$ and each $\alpha>0$, we have*

(i) $\mathsf{Op}^{+n}_{q_1}\Leftrightarrow \mathsf{p}_{n+1}$,

(ii) $\mathsf{Op}^{+n}_{(m+1,x)}\Leftrightarrow (\mathsf{Op}^{+(n+1)}_{x}\mathsf{p}_{n+1})^{m+1}$ *and* $\mathsf{Op}^{+n}_{(\gamma+m+1,x)}\Leftrightarrow (\mathsf{Op}^{+(n+1)}_{x}\mathsf{p}_{n+1})^{\gamma+m}$,

(iii) $\mathsf{Op}^{+n}_{(\gamma,x)}\Leftrightarrow (\mathsf{p}_{n+1}\mathsf{Op}^{+n}_{(1,x)})^{\gamma}$,

(iv) $\mathsf{Op}^{+n}_{(1+\alpha,y^-)}\Leftrightarrow (\mathsf{Op}^{+(n,n+1)}_{y^-}\mathsf{p}_{n+1})^{1+\alpha}$

(v) $\mathsf{Op}^{+n}_{\langle x_1,\dots,x_k\rangle}\Leftrightarrow \mathsf{Op}^{+n}_{x_1}\circ\dots\circ\mathsf{Op}^{+n}_{x_k}$,

and for all m,n with $0\le m<n$, we have

(iii)' $\mathsf{Op}^{+(m,n)}_{(\gamma,v)^-}\Leftrightarrow (\forall\alpha<\gamma)\mathsf{p}_{m+1}(\mathsf{Op}^{+n}_{(1,v)})^{1+\alpha}$,

(vi)' $\mathsf{Op}^{+(m,n)}_{(\alpha+1,y^-)^-}\Leftrightarrow \mathsf{Op}^{+(m,n+1)}_{y^-}\mathsf{p}_{n+1}\circ\mathsf{Op}^{+n}_{(\alpha,y^-)}$,

(v)' $\mathsf{Op}^{+(m,n)}_{\langle y_1,\dots,y_k\rangle^-}\Leftrightarrow \mathsf{Op}^{+(m,n)}_{y_1^-}\circ\mathsf{Op}^{+n}_{y_2}\circ\dots\circ\mathsf{Op}^{+n}_{y_k}$.

Proof (i) is by definition, (ii) and (iii) are by Lemma III.6.20, and (v) is by Lemma III.6.5 (ii). To show (iv), we first observe, using the definitions of $\mathsf{Op}^{+n}_{(1,y^-)}$ and $\mathsf{Op}^{+(n,n+1)}_{y^-}$ (i.e. Definitions III.6.1, III.6.3) and Lemma III.6.5 (iii), that

$$\mathsf{Op}^{+n}_{(1,y^-)}\Leftrightarrow(\forall\alpha\lhd o(y))(\mathsf{p}_{n+1}\mathsf{Op}^{+n}_{(1,y^-)[\alpha]})\Leftrightarrow(\forall\alpha\lhd o(y))(\mathsf{p}_{n+1}\mathsf{Op}^{+(n+1)}_{y[\alpha]}\mathsf{p}_{n+1})\Leftrightarrow$$
$$(\forall\alpha\lhd o(y))(\mathsf{p}_{n+1}\mathsf{Op}^{+(n+1)}_{y[\alpha]})\mathsf{p}_{n+1}\Leftrightarrow\mathsf{Op}^{+(n,n+1)}_{y^-}\mathsf{p}_{n+1}.$$

Now we show, using Theorem I.4.2, that $A(\alpha):=\mathsf{Op}^{+n}_{(\alpha,y^-)}(\breve{\mathsf{T}})\leftrightarrow(\mathsf{Op}^{+n}_{(1,y^-)})^{\alpha}(\breve{\mathsf{T}})$, for $\alpha\rhd 0$. (iv) then follows. $A(1)$ trivially, holds. If $1<\alpha=\alpha'+1$, then $\mathsf{Op}^{+n}_{(\alpha,y^-)}$ iff $\mathsf{Op}^{+n}_{(1,y^-)}\circ\mathsf{Op}^{+n}_{(\alpha',y^-)}$, and $A(\alpha)$ follows using the I.H. If $\alpha=:\gamma$ is a limit, then $\mathsf{Op}^{+n}_{(\gamma,y^-)}$ iff $(\forall\xi\lhd\gamma)\mathsf{p}_{n+1}\mathsf{Op}^{+n}_{(1+\xi,y^-)}$. The I.H. implies that $\mathsf{Op}^{+n}_{(\gamma,y^-)}$ iff $(\forall\xi\lhd\gamma)\mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{(1,y^-)})^{1+\xi}$. As $(\mathsf{Op}^{+n}_{(1,y^-)})^{1+\xi}$ is Π^1_{n+2}, we have that for all $0\lhd\xi\lhd\gamma$, $\mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{(1,y^-)})^{\xi}\Rightarrow(\mathsf{Op}^{+n}_{(1,y^-)})^{\xi}$, and $\mathsf{Op}^{+n}_{(1,y^-)})^{\xi+1}\Rightarrow\mathsf{p}_{n+1}(\mathsf{Op}_{(1,y^-)})^{\xi}$, hence $A(\gamma)$ follows.

For the next clauses, recall that $\mathsf{Op}^{+(m,n)}_{x^-}\Leftrightarrow(\forall\alpha\lhd o(x))\mathsf{p}_{m+1}\mathsf{Op}^{+n}_{x[\alpha]}$. For (iii)' observe that

$$\mathsf{Op}^{+(m,n)}_{(\gamma,v)^-}\Leftrightarrow(\forall\alpha\lhd o(v))\mathsf{p}_{m+1}\mathsf{Op}^{+n}_{(1+\alpha,v)}\Leftrightarrow(\forall\alpha\lhd o(v))\mathsf{p}_{m+1}(\mathsf{p}_{n+1}\mathsf{Op}^{+n}_{(1,v)})^{1+\alpha}\Leftrightarrow$$
$$(\forall\alpha\lhd o(v))\mathsf{p}_{m+1}\mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{(1,v)})^{1+\alpha}\Leftrightarrow(\forall\alpha\lhd o(v))\mathsf{p}_{m+1}\mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{(1,v)})^{1+\alpha}.$$

(iv)' Let $(\beta{+}1, y^-) = o(y) = \gamma$. Then, $\mathsf{Op}^{+(m,n)}_{(\beta+1,y^-)^-} \Leftrightarrow (\forall\alpha \lhd \gamma)\mathsf{p}_{m+1}\mathsf{Op}^{+n}_{(\beta+1,y^-)[\alpha]}$. As $\mathsf{Op}^{+n}_{(\beta+1,y^-)[\alpha]} \Leftrightarrow \mathsf{Op}^{+n}_{(1,y^-)[\alpha]} \circ \mathsf{Op}^{+n}_{(\beta,y^-)}$, the claim follows.

(v)' Assume that $k > 1$ and $x = \langle x_1, \ldots, x_k\rangle$. Let $\gamma := o(x)$ and $y =: \langle x_2, \ldots, x_k\rangle$. Recall that $x[\alpha] = x_1[\alpha] * y$. By definition, $\mathsf{Op}^{+(m,n)}_{x^-} \Leftrightarrow (\forall\alpha \lhd \gamma)\mathsf{p}_{m+1}\mathsf{Op}^{+n}_{x[\alpha]}$. As $\mathsf{Op}^{+n}_{x[\alpha]} \Leftrightarrow \mathsf{Op}^{+n}_{x_1[\alpha]} \circ \mathsf{Op}^{+n}_y$, we have $\mathsf{Op}^{+(m,n)}_{x^-} \Leftrightarrow (\forall\alpha\lhd\gamma)\mathsf{Op}^{+n}_{x_1[\alpha]} \circ \mathsf{Op}^{+n}_y \Leftrightarrow \mathsf{Op}^{+n}_{x_1^-} \circ \mathsf{Op}^{+n}_y$. $\square$

III.7 Modular ordinal analysis at work again

In this section, we extend what we did in Section II.5 to the general case. Again, we fix $\breve{\mathsf{T}} := (\mathsf{ACA})$ and $g(\alpha) := \omega^{1+\alpha}$, and prove $\mathsf{T}^\epsilon \vdash \bigwedge_{n<N_0}(\forall x \in Q^*_{N_0})\mathsf{Prv}_{n+1}(x)$, and obtain $\mathsf{T}^\epsilon \vdash (\forall x \in Q_{N_0})\mathsf{Prv}_0(x)$ as an easy corollary.

III.7.1 Lifting "Op_x proves H_{x^H}"

Let us lift all the definitions to the general case. This is very canonic, and mostly the definitions look the same. However, keep in mind that the underlying set of names Q_{N_0} is now bigger, and that this change affects $\breve{\mathsf{T}}_x$ (e.g. $\breve{\mathsf{T}}_x \to x \in Q_{N_0}$).

Definition III.7.1. *For $x \in Q$,* $x^h := \begin{cases} (x{+}1)^H & : x < (\omega, q_0), \\ x^H & : \textit{else.} \end{cases}$

With Lemma III.5.12 at hand, we can think of g_{x_h} as $(H_{x^H} \circ \mathsf{it})(g) = f_{x^H}$ for $f := \mathsf{it}(g)$.

Definition III.7.2. $\mathcal{C}_x := \{\alpha : [(\deg(x) \le 1 \wedge \mathsf{Wo}_\lhd(g_{x^h}(\alpha))] \vee [\deg(x) \ge 2 \wedge \breve{\mathsf{T}}_{x(\alpha)}]\}$.

Next, we define when T_x proves g_{x^h}, when Op_x proves H_{x^H}, and when Op^{m+1}_x proves $H^{m+1}_{x^*}$, where again, $x^* := x^H{+}\mathsf{corr}(x)$. The first two notions are as before, and the third extends Op^+_x proves $H^+_{x^*}$: if for $0 \le i \le m$, $\mathsf{Op}^{+i}_{(1,y_{m-i})}$ proves $H^{+i}_{(1,y_{m-i})^H}$, then

$$\mathsf{Op}^{+(m+1)}_x(\mathsf{Op}^{+m}_{(1,y_0)} \circ_1 \ldots \circ_1 \mathsf{Op}^{+0}_{(1,y_m)}) \text{ proves } H^{+(m+1)}_{x^*}[H^{+m}_{(1,y_0)^H}, \ldots, H^{+0}_{(1,y_m)^H}].$$

The definition below is worded differently, but the way to think of Op^{m+1}_x proves $H^{m+1}_{x^*}$ is as explained above. However, this only serves as motivation since for $z := L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_m))$, Op_z iff $\mathsf{Op}^{+(m+1)}_x(\mathsf{Op}^{+m}_{(1,y_0)} \circ_1 \ldots \circ_1 \mathsf{Op}^{+0}_{(1,y_m)})$, and by Corollary III.5.14, H_z is "about equal" to $H^{+(m+1)}_{x^*}[H^{+m}_{(1,y_0)^H}, \ldots, H^{+0}_{(1,y_m)^H}]$ (cf. Lemma III.7.4).

Definition III.7.3. *We fix the following formulas.*

$$\begin{aligned}\mathsf{Prv}_0(x) &:= \breve{\mathsf{T}}_x \to \forall\alpha[\mathsf{Wo}_\lhd(\alpha) \wedge \mathsf{TI}_\lhd(\mathcal{C}_x,\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))],\\ \mathsf{Prv}_1(x) &:= \forall y[\mathsf{prv}_0(y) \to \mathsf{Prv}_0(x\circ y)],\\ \mathsf{Prv}_{n+2}(x) &:= \forall y[\mathsf{prv}_{n+1}(1,y) \to \mathsf{Prv}_{n+1}(1,x\circ y)].\end{aligned}$$

Further, $\mathsf{prv}_n(x) := \forall X \mathsf{Prv}_n(x)\restriction X$*.*

We say that Op_x *proves* H_{x^H} *if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_1(x)$*, and that* $\mathsf{Op}_x^{+(n+1)}$ *proves* $H_{x^*}^{+(n+1)}$*, if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_{n+2}(x)$*.*

Again, $\breve{\mathsf{T}}_u \to u \in Q_{N_0}$ (cf. Definition III.6.1 and Lemma III.6.2), and $\mathsf{Prv}_0(u)$ is trivially true if $u \notin Q_{N_0}$. So $\mathsf{Prv}_1(x)$ iff $\forall y[x\circ y \in Q_{N_0} \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(x\circ y)]$, and $\mathsf{Prv}_{m+2}(x)$ iff $\forall y[x\circ y \in Q_{N_0-m}\mathsf{prv}_{m+1}(1,y) \to \mathsf{Prv}_{m+1}(1,x\circ y)]$. Often, we use these equivalent forms to focus on the non-trivial instances of these definitions.

The next lemma partially unfolds the definition of $\mathsf{Prv}_{n+1}(x)$. To do so, we use for a possibly empty list $\vec{y} = y_0,\ldots,y_n$ the term $t_n(x,\vec{y}) := L(x\circ_1(1,y_0)\circ_1\ldots\circ_1(1,y_{n-1}))$. In case that $n=0$, $t_0(x) = x$.

Lemma III.7.4. T^ϵ *proves the following.*

$$\mathsf{Prv}_{n+1}(x) \leftrightarrow \forall\vec{y},y[z = L(x\circ_1(1,y_0)\circ_1\ldots\circ_1(1,y_{n-1}))\circ y \wedge \bigwedge_{i<n}\mathsf{prv}_{n-i}(1,y_i) \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(z)].$$

Proof By meta-induction on n. If $n=0$, the right hand side is the definition of $\mathsf{Prv}_1(x)$. To show the induction step, let $t_n(x,y_0,\ldots,y_{n-1}) := L(x\circ_1(1,y_0)\circ_1\ldots\circ_1(1,y_{n-1}))$. By definition,

$$\mathsf{Prv}_{n+2}(x) \leftrightarrow \forall y_0[\mathsf{prv}_{n+1}(y_0) \to \mathsf{Prv}_{n+1}(1,x\circ y_0)].$$

By I.H.,

$$\mathsf{Prv}_{n+1}(1,x\circ y_0) \leftrightarrow \forall y_1,\ldots,y_n,y[t_n((1,x\circ y_0),y_1,\ldots,y_n)\circ y \wedge \bigwedge_{1\le i<n+1}\mathsf{prv}_{n+1-i}(1,y_i) \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(t_n((1,x\circ y_0),y_1,\ldots,y_n)\circ y)].$$

Since $t_n(x\circ_1(1,y_0),y_1,\ldots,y_n) = t_{n+1}(x,y_0,y_1,\ldots,y_{n+1})$, the claim follows. $\square$

Looking at the definition of $\mathsf{Prv}_0(z)$, we also have the following.

Corollary III.7.5. T^ϵ *proves the following.*

$$\mathsf{Prv}_{n+1}(x) \leftrightarrow \forall \vec{y}[z = L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_{n-1})) \wedge \bigwedge_{i<n} \mathsf{prv}_{n-i}(1, y_i) \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_1(z)].$$

Hence $\mathsf{Prv}_{m+2}(x)$ has the following alternative characterization, which justifies the aforementioned motivation. If $z = L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_m))$, then $\bigwedge_{i \leq m} \mathsf{prv}_{m+1-i}(y_i)$ implies $\mathsf{Prv}_1(z)$, i.e. Op_z proves H_{z^H}.

Next, we check that $(\forall x \in Q^*_{N_0-n})\mathsf{Prv}_{n+1}(x)$ can be proved using Theorem I.4.2.

Lemma III.7.6. $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_{n+1}(x) \vee ((\mathsf{ACA}) \wedge \mathsf{Wf}_{\rightsquigarrow^*}(x))$.

Proof We work informally in T^ϵ and assume $\neg\mathsf{Prv}_{n+1}(x)$. The above lemma yields that there are $z, \vec{y}, y$, so that $z = L(x \circ_1 (1, y_0) \circ_1 \ldots \circ_1 (1, y_{n-1})) \circ y \in Q^*_{N_0}$ and $\neg\mathsf{Prv}_0(z)$. Further, $\neg\mathsf{Prv}_0(z)$ implies $\breve{\mathsf{T}}_z$ which in turn implies (ACA) and $\mathsf{Wf}_{\rightsquigarrow^*}(z)$. And finally, over ACA_0, $\mathsf{Wf}_{\rightsquigarrow^*}(z)$ implies $\mathsf{Wf}_{\rightsquigarrow^*}(x)$: By Lemma III.4.19 which is clearly already provable in ACA_0, we have that $x' \rightsquigarrow^* x'' \rightsquigarrow^* x$ implies $t_n(x', \vec{y}) \circ y \rightsquigarrow^* t_n(x'', \vec{y}) \rightsquigarrow^* t_n(x, \vec{y}) = z$. Hence, an infinite descending chain $x_0 \rightsquigarrow *x_1 \rightsquigarrow \ldots$ with elements in $\{x' : x' \rightsquigarrow^* x\}$ gives rise to an infinite descending chain $t_n(x_0, \vec{y}) \circ y \rightsquigarrow^* t_n(x_1, \vec{y})$ with elements in $\{z' : z' \rightsquigarrow^* z\}$. □

As discussed on page 64, if e.g. $\mathsf{ACA}_0 \vdash A \wedge b \to \mathsf{Prv}_n(x)$ (where $b = \forall X B\restriction X$), then we also have $\mathsf{T}^\epsilon \vdash a \wedge \forall X b\restriction X \to \mathsf{prv}_n(x)$ and $\mathsf{ACA}_0 \vdash a \wedge b \to \mathsf{prv}_n(x)$. Again, we refer to this as the "small variant" of $\mathsf{ACA}_0 \vdash A \wedge b \to \mathsf{Prv}_n(x)$. For instance, we have the following which we will tacitly use in the sequel.

Lemma III.7.7. *The following is provable in* ACA_0*:*

(i) $x \circ y \in Q^*_{N_0} \wedge \mathsf{prv}_1(x) \wedge \mathsf{prv}_0(y) \to \mathsf{prv}_0(x \circ y)$.

(ii) $x \circ y \in Q^*_{N_0-n-1} \wedge \mathsf{prv}_{n+2}(x) \wedge \mathsf{prv}_{n+1}(1, y) \to \mathsf{prv}_{n+1}(1, x \circ y)$.

(iii) $x \circ y \in Q^*_{N_0-n} \wedge \mathsf{Prv}_{n+1}(x) \wedge \mathsf{prv}_{n+1}(y) \to \mathsf{Prv}_{n+1}(x \circ y)$.

Proof (i) and (ii) are the "small variants" of $x \circ y \in Q^*_{N_0} \wedge \mathsf{Prv}_1(x) \wedge \mathsf{prv}_0(y) \to \mathsf{Prv}_0(x \circ y)$ and $x \circ y \in Q^*_{N_0-n-1} \wedge \mathsf{Prv}_{n+2}(x) \wedge \mathsf{prv}_{n+1}(1, y) \to \mathsf{Prv}_{n+1}(1, x \circ y)$, which hold by definition of $\mathsf{Prv}_1(x)$ and $\mathsf{Prv}_{n+2}(x)$, respectively. (iii) $\mathsf{Prv}_{n+1}(x \circ y)$ holds, if $x \circ y \circ z \in Q^*_{N_0-n}$ and $\mathsf{prv}_n(z)$ imply $\mathsf{Prv}_n(x \circ y \circ z)$. To verify the claim, assume $x \circ y \circ z \in Q^*_{N_0-n}$ and $\mathsf{prv}_{n+1}(z)$, and further $\mathsf{Prv}_{n+1}(x)$ and $\mathsf{prv}_{n+1}(y)$. By (i), $\mathsf{prv}_{n+1}(y)$ and $\mathsf{prv}_n(z)$ yield $\mathsf{prv}_n(y \circ z)$, and $\mathsf{Prv}_n(x \circ y \circ z)$ follows from $\mathsf{Prv}_n(x)$. □

Again, all we need to know about the interplay of $\cdot[\alpha]$, $\cdot(\alpha)$ and $\cdot^H$ is collected below (we will use (i) in the proof of Lemma III.7.10 (ii), and (ii) in the proof of Lemma III.7.12).

Lemma III.7.8. *Let $z \in Q^*$. Then we have the following.*

(i) If $\deg(z) = 1$ *and* $o(z) = \gamma$*, then* $g_{z^H[\alpha]} \trianglelefteq g_{(z[\alpha])^H}$*,*

(ii) if $\deg(z) > 1$*, then* $g_{z^H[\alpha]} \trianglelefteq g_{(z(\alpha))^H}$*.*

Proof By Lemma III.4.24, we have in case (i) $z^H[\alpha] \rightsquigarrow^* (z[\alpha])^H$, and in case (ii) $z^H[\alpha] \rightsquigarrow^* z(\alpha))^H$. Thus the claim follows by Lemma III.5.1. □

III.7.2 A sketch of the proof

In this subsection, we sketch the proof of $\mathsf{T}^\epsilon \vdash \bigwedge_{n<\mathsf{N}_0}(\forall x \in Q^*_{N_0})\mathsf{Prv}_{n+1}(x)$ given in the next subsection. Again, we neglect the difference between $\mathsf{Prv}_m(x)$ and $\mathsf{prv}_m(x)$ ($m \leq N_0$), and pretend that for $m \leq N_0$, $\mathsf{Prv}_m(x) \wedge \mathsf{Prv}_m(y)$ implies $\mathsf{Prv}_m(x \circ y)$.

Recall that in the case $N_0 = 2$, the assumption $\mathsf{Prv}_2(q_1)$ allowed us to prove $\forall\beta\mathsf{Prv}_2(1{+}\beta, x)$ for all (the only) names $x \in Q_0 = \{q_0\}$. In a next step, we then proved $(\forall x \in Q^*_2)\mathsf{Prv}_1(x)$ by induction on $\rightsquigarrow^*$. For $x =_{NF} (1, y) \circ z$, the I.H. and $z \rightsquigarrow^* x$ gave us $\mathsf{Prv}_1(z)$, and $\mathsf{Prv}_2(y)$ and $\mathsf{Prv}_1(1, q_0)$ gave us $\mathsf{Prv}_1(1, y)$. Together, $\mathsf{Prv}_1(1, y)$ and $\mathsf{Prv}_1(z)$ implied $\mathsf{Prv}_1(x)$.

Now we briefly say how to extend the proof, more details are given in following paragraphs. First, observe that Lemma II.5.17, which allowed us to move from $\mathsf{Prv}_2(q_1)$ to $\forall\beta\mathsf{Prv}_2(1{+}\beta, q_0)$, is a special case ($v = q_0$) of the "one-up variant" of Lemma II.5.16, where Prv_1 and prv_1 are replaced by Prv_2 and prv_2, respectively. And further, the "one-up variant" of Lemma II.5.17 (ii) is an easy consequence of (i), and (i) is a consequence of Lemma II.5.16 and Lemma II.5.15. Moreover, as we will see below, Lemma II.5.15 implies its own 'one-up variant", which together with the "one-up variant" of Lemma II.5.16 implies a "two-up variant" of Lemma II.5.16 , which then readily yields $\mathsf{Prv}_3(1, q_0)$, and so on.

Let us elaborate on the above comment. For a start, we assume the following, where (a_m) and (a'_m) are the "m-up" variants of Lemma II.5.16, (b_m) is the "m-up" variant of Lemma II.5.15, and, as in the corresponding sketch for $N_0 = 2$, $(c\ \ )$ hides technical details (cf. Lemma III.7.10).

(a_m) if $\deg(x) = 1$, then $(\forall\alpha \triangleleft o(x))\mathsf{Prv}_{m+1}(x[\alpha])$ implies $\mathsf{Prv}_{m+1}(x)$,

(a'_m) if $\deg(x) = 1$, then $\mathsf{Prv}_{m+1}(1, v)$ implies $(\forall\alpha\mathsf{Prv}_{m+1}(1{+}\alpha, v)$,

(b_m) if $\deg(x) \geq 2$, then $\forall\alpha\mathsf{Prv}_{m+1}(x(\alpha))$ implies $\mathsf{Prv}_{m+1}(x)$,

$(c\ \ )$ if $\deg(x) = 1$, then $\breve{\mathsf{T}}_x \wedge (\forall\alpha \triangleleft o(x))\mathsf{Prv}_0(x[\alpha]) \rightarrow \mathsf{Prog}_\triangleleft(\mathcal{C}_x)$.

From the above assumptions we continue as follows: $\mathsf{Prv}_1(q_1)$ and $\mathsf{Prv}_2(q_1)$ are obtained analogously to the case $N_0 = 2$, and $\mathsf{Prv}_{m+2}(q_1)$ is by (a'_m): if $\mathsf{Prv}_{m+1}(1, y)$, then (a'_m) yields $\forall\alpha\mathsf{Prv}_{m+1}(1{+}\alpha, y)$, which further, together with $\mathsf{Prv}_{m+1}(q_1)$ yields $\forall\alpha\mathsf{Prv}_{m+1}((1{+}\alpha, y){+}1)$, that is, $\forall\alpha\mathsf{Prv}_{m+1}((1, y{+}1)(\alpha))$, and now $\mathsf{Prv}_{m+1}((1, y{+}1)$ follows by (b_m).

While in the case $N_0 = 2$, we first showed $(\forall x \in Q^*_1)\mathsf{Prv}_2(x)$ and only then proved $(\forall x \in Q^*_2)\mathsf{Prv}_1(x)$, we show in the general case $(\forall x \in Q^*_{n+1})\mathsf{Prv}_{N_0-n}(x)$ by meta-induction on $n < N_0$. Since $\mathsf{Prv}_{N_0-n}(x)$ implies $x \in Q_{n+1}$, it suffices to prove $(\forall x \in Q^*)\mathsf{Prv}_{N_0-n}(x)$; the possible cases are discussed below.

(i) $\deg(x) = 1 \wedge x \neq q_1$. By I.H., $(\forall\alpha \lhd o(x))\mathsf{Prv}_{N_0-n}(x[\alpha])$, and $\mathsf{Prv}_{N_0-n}(x)$ is by (a_{N_0-n-1}).

(ii) $\deg(x) > 1$. Then $x =_{NF} (1, y) \circ z$. As $z \rightsquigarrow^* x$ by Lemma III.4.19 (iv), the I.H. yields $\mathsf{Prv}_{m+1}(z)$. If $n = 0$, then $(1, y) \in Q_1$, thus $y = q_0$. Hence $\mathsf{Prv}_{N_0}(q_1)$. If $n > 0$, then $\mathsf{Prv}_{N_0-(n-1)}(y)$ by the meta-I.H. Together with $\mathsf{Prv}_{N_0-n}(1, q_0)$ we obtain $\mathsf{Prv}_{N_0-n}(1, y)$. Now $\mathsf{Prv}_{N_0-n}(1, y)$ and $\mathsf{Prv}_{N_0-n}(z)$ imply $\mathsf{Prv}_{N_0-n}(x)$.

Finally, we discuss where we get the assumptions (a_m), (a'_m) and (b_m) from. (b_1) is shown as (b) in the case $N_0 = 2$, and (b_{m+1}) follows readily form (b_m): if $\deg(x) > 1$ and $\forall\alpha\mathsf{Prv}_{m+2}(x(\alpha))$, then $\mathsf{Prv}_{m+1}(1, y)$ yields $\forall\alpha\mathsf{Prv}_{m+1}(z(\alpha))$ for $z := (1, x \circ y)$, which by (b_m) yields $\mathsf{Prv}_{m+1}(z)$. This shows $\mathsf{Prv}_{m+2}(x)$.

Also (a_1) is obtained as in the case $N_0 = 2$, and (a'_m) follows readily form (a_m). Further, (a_{m+1}) is by (a'_m) and (b_m). Again, we observe a characteristic two-step approximation which works almost literally as in the case $N_0 = 2$.

III.7.3 Proof of the main result for the general case

Next, we observe our modular approach at work in the general case. The following results are proved exactly as the corresponding results (Lemma II.5.11 and II.5.12) in Section II.5.

Lemma III.7.9. ACA_0 *proves the following.*

(i) $\mathsf{Prv}_0(q_0)$,

(ii) $x \in Q_{N_0} \wedge \breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \wedge \mathsf{Wo}_{\lhd}(\alpha) \to \mathsf{Wo}_{\lhd}(g_{x^h}(\alpha))$.

Lemma III.7.10. ACA_0 *proves the following.*

(i) $x \in Q_{N_0} \wedge \breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \to \mathsf{Prog}_{\lhd}(\mathcal{C}_{x+1})$.

(ii) $x \in Q^*_{N_0} \wedge \deg(x) = 1 \wedge o(x) = \gamma \wedge (\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\alpha]) \wedge \breve{\mathsf{T}}_x \to \mathsf{Prog}_{\lhd}(\mathcal{C}_x)$.

Proof: Literally(!) as the proof of the corresponding Lemma II.5.13, expect that we now refer to Lemma III.6.5) (i) instead of Lemma II.4.4 (ii), and to Lemma III.7.8 instead of Lemma II.5.10.

Again, as a consequence of Lemma III.7.10 (i), we obtain that p_1 proves it.

Lemma III.7.11. $\mathsf{ACA}_0 \vdash \mathsf{Prv}_1(q_1)$.

Below, we lift Lemma II.5.15 (which corresponds to (b_n) in the sketch) to the general case.

Lemma III.7.12. *For each $n < N_0$,*

$$\mathsf{ACA}_0 \vdash (\forall x \in Q^*_{N_0})[\deg(x) > 1 \wedge \forall\alpha \mathsf{prv}_{n+1}(x(\alpha)) \to \mathsf{Prv}_{n+1}(x)].$$

Proof By meta-induction on $n < N_0$. The case $n = 0$ is literally as the proof of the corresponding Lemma II.5.15, but referring now to Lemma III.7.8 instead of Lemma II.5.10. For the induction step, assume that $n > 0$, and that the claim holds for $n{-}1$. Assume that $x \in Q^*_{N_0}$ with $\deg(x) > 1$ and $\forall\alpha \mathsf{prv}_{n+1}(x(\alpha))$, and aim for $\mathsf{Prv}_{n+1}(x)$. For that, further assume that $x \circ y \in Q^*_{N_0}$ and $\mathsf{prv}_n(1, y)$, and aim for $\mathsf{Prv}_n(1, x \circ y))$. $\mathsf{prv}_{n+1}(x(\alpha))$ and $\mathsf{prv}_n(1, y)$ imply $\mathsf{prv}_n((1, x(\alpha) \circ y)$, that is, $\mathsf{prv}_n(1, x \circ y)(\alpha))$, as $\deg(x) \geq 2$ (cf. Lemma III.4.16). Hence we have $\forall\alpha \mathsf{prv}_n(1, x \circ y)(\alpha))$, and the I.H. yields $\mathsf{Prv}_n(1, x \circ y)$. □

The next result corresponds to Lemma II.5.16 and to (a_n) of the sketch.

Lemma III.7.13. *For each $n < N_0$, ACA_0 proves the following: for each $x \in Q_{N_0}$ with $\deg(x) = 1 \wedge o(x) = \delta_0{+}\gamma$, and each $(1, v) \in Q^*_{N_0}$, then*

(i) $(\forall\alpha \lhd \gamma)\mathsf{prv}_{n+1}(x[\delta_0{+}\alpha]) \to \mathsf{Prv}_{n+1}(x) =: C_1(n)$,

(ii) $\mathsf{Prv}_{n+1}(1, v) \wedge \mathsf{prv}_{n+1}(1, v) \to \forall\alpha \mathsf{Prv}_{n+1}(1{+}\alpha, v) =: C_2(n)$.

Proof First note that (ii) follows using (i) by induction on α (in the sense of Corollary I.4.3) exactly as in the case $N_0 = 2$. Hence, it suffices to show (i), which is done by meta-induction on $n < N_0$. The case $n = 0$ is literally as the proof of the corresponding Lemma II.5.16 (i), but referring to Lemma III.7.10 (ii) instead of Lemma III.7.10 (ii). Next, we consider the induction step. It is assumed that $n > 0$, and that (i) and (ii) hold for $n{-}1$, that is, $C_1(n{-}1)$ and $C_2(n{-}1)$. We show that (i) holds for n, i.e. that $C_1(n)$.

Assume that $x \in Q^*_{N_0}$ with $\deg(x) = 1$ and $o(x) = \delta_0{+}\gamma$ and $(\forall\alpha{\lhd}\gamma)\mathsf{prv}_{n+1}(x[\delta_0{+}\alpha])$, and aim for $\mathsf{Prv}_{n+1}(x)$. For that, further assume that $z := (1, x \circ y) \in Q^*_{N_0}$ and $\mathsf{prv}_n(1, y)$, and aim for $\mathsf{Prv}_n(z)$. Thereto, let δ_1 so that for each β, $x[\beta] \circ y = (x{\circ}y)[\delta_1{+}\beta]$ and $o(z) = o(x{\circ}y) = \delta_1{+}\gamma$ (cf. Lemma III.4.16). Recall that by Lemma

III.4.12 (iv), $(1,(x\circ y)^-)[\alpha] = (1,(x\circ y)[\alpha])$. The assumptions $(\forall\alpha\lhd\gamma)\mathsf{Prv}_{n+1}(x[\alpha])$ and $\mathsf{prv}_n(1,y)$ yield for each $\alpha<\gamma$, $\mathsf{Prv}_n(1,x[\alpha]\circ y)$, that is $\mathsf{Prv}_n((1,(x\circ y)^-)[\delta_1{+}\alpha])$. Hence the I.H. yields $\mathsf{Prv}_n(1,(x\circ y)^-)$, and its "small variant" yields $\mathsf{prv}_n(1,(x\circ y)^-)$. So by (ii), $\forall\beta\mathsf{prv}_n(1{+}\beta,(x\circ y)^-)$, that is, $\forall\beta\mathsf{prv}_n(z(\beta))$. Finally, $\mathsf{Prv}_n(z)$ is by Lemma III.7.12. □

The next result lifts Lemma II.5.18 to the general case.

Lemma III.7.14. *For all* $n<N_0$*,* $\mathsf{ACA}_0\vdash\mathsf{Prv}_{n+1}(q_1)$*.*

Proof By meta-induction on n. The case $n=0$ is by Lemma III.7.11. For the induction step, assume that the claim holds for n. To show $\mathsf{Prv}_{n+2}(q_1)$, assume $y{+}1\in Q_{N_0}$ and $\mathsf{prv}_{n+1}(1,y)$, and aim for $\mathsf{Prv}_{n+1}(x)$ for $x:=(1,y{+}1)$. Note that $\deg(x)>1$. Once we know that $\forall\alpha\mathsf{prv}_{n+1}(x(\alpha))$, the claim is by Lemma III.7.12. By the small variant of Lemma III.7.13 (ii), $\mathsf{prv}_{n+1}(1,y)$ yields $\forall\alpha\mathsf{prv}_{n+1}(1{+}\alpha,y)$. Since $x(\alpha)=x[\alpha]{+}1=(1{+}\alpha,y){+}1$, $\mathsf{prv}_{n+1}(1{+}\alpha,y)$ and $\mathsf{prv}_{n+1}(q_1)$ yield $\mathsf{prv}_{n+1}((1{+}\alpha,y){+}1)$, we also have $\forall\alpha\mathsf{prv}_{n+1}(x(\alpha))$. □

Putting the pieces together yields a proof of the main result of part I. It generalizes Theorem III.7.15.

Theorem III.7.15. *For all* $n<N_0$*,* $\mathsf{ACA}_0\vdash(\forall x\in Q^*_{n+1})\mathsf{Prv}_{N_0-n}(x)$*.*

Proof By meta-induction on $n<N_0$. If $n=0$ and $x\in Q^*_1$, then $x=(1{+}\beta,q_0)$. As we have $\mathsf{Prv}_{N_0}(q_1)$ by Lemma III.7.14, and thus also $\mathsf{prv}_{N_0}(q_1)$, $\mathsf{Prv}_{N_0}(1{+}\beta,q_0)$ is by Lemma III.7.13 (ii).

For the induction step, assume $n{+}1<N_0$ and $\mathsf{ACA}_0\vdash(\forall x\in Q^*_{n+1})\mathsf{Prv}_{N_0-n}(x)$. We show $\mathsf{ACA}_0\vdash(\forall x\in Q^*_{n+2})\mathsf{Prv}_{N_0-n-1}(x)$ by induction on $\rightsquigarrow^*\restriction Q^*_{n+2}$ (in the sense of Theorem I.4.2). We consider the following possible cases.

(i) $x=y{+}1$. If $x=q_1$, $\mathsf{Prv}_{N_0-n-1}(q_1)$ is by Lemma III.7.14. Else, we have $\mathsf{prv}_{N_0-n-1}(y)$ by I.H. Together with $\mathsf{Prv}_{N_0-n-1}(q_1)$, this yields $\mathsf{Prv}_{N_0-n-1}(x)$.

(ii) $\deg(x)=1\wedge o(x)=\gamma$. By I.H., $(\forall\alpha\lhd\gamma)\mathsf{prv}_{N_0-n-1}(x[\alpha])$, and $\mathsf{Prv}_{N_0-n-1}(x)$ is by Lemma III.7.13 (i).

(iii) $\deg(x)>1$. Then there are y,z with $\deg(y)>0$ so that $x=_{NF}(1,y)\circ z$. As $z\rightsquigarrow^* x$ by Lemma III.4.19 (iv), the I.H. yields $\mathsf{prv}_{N_0-n-1}(z)$. As $(1,y)\in Q^*_{n+2}$ and $\deg(y)>0$, we have $y\in Q^*_{n+1}$, and $\mathsf{prv}_{N_0-n}(y)$ is by the meta-I.H. Together with $\mathsf{Prv}_{N_0-n-1}(1,q_0)$ we obtain $\mathsf{Prv}_{N_0-n-1}(1,y)$. Finally, $\mathsf{Prv}_{N_0-n-1}(1,y)$ and $\mathsf{Prv}_{N_0-n-1}(z)$ imply $\mathsf{Prv}_{N_0-n-1}(x)$.

□

In particular, we have that for each name $x\in Q$, T_x proves g_{x^h}.

Corollary III.7.16. $\mathsf{T}^\epsilon \vdash (\forall x \in Q_{N_0})\mathsf{Prv}_0(x)$.

Proof Let $x \in Q_{N_0}$. We have $\mathsf{ACA}_0 \vdash \mathsf{Prv}_0(q_0)$, hence also $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(q_0)$, which entails $\mathsf{T}^\epsilon \vdash \mathsf{prv}_0(q_0)$. Togehter with $\mathsf{Prv}_1(x)$, this implies $\mathsf{Prv}_0(x)$. $\square$

The corollary immediately provides lower bounds for the proof-theoretic ordinal of a theory of the form $\breve{\mathsf{T}}_x$ or $\breve{\mathsf{T}}_x + (\mathsf{I_N})$ (cf. the discussion following Corollary II.5.20.

Below, we list a few instances of the above corollary. In particular, it is confirmed what we claimed already at the end of Section II.5, namely that $|\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA}_0)| \geq \vartheta\Omega^\Omega$. The presentation of the ordinals in the form $\vartheta\gamma$ is due to Corollary IV.5.13. Further, $\Omega_0 := 1$ and $\Omega_{n+1} := \Omega^{\Omega_n}$.

Example III.7.17.

(i) $|\mathsf{p}_3(\mathsf{ACA}_0)| \geq (\mathsf{It}_3, \mathsf{It}, \mathsf{it}, g, \omega) = (\mathsf{It}^\omega(\mathsf{it}), g, 0) = \vartheta\Omega^\omega$ *(small Veblen number),*

(ii) $|\mathsf{p}_3(\mathsf{ACA}_0) + (\mathsf{I_N})| \geq (\mathsf{It}_3, \mathsf{It}, \mathsf{it}, g, \varepsilon_0) = (\mathsf{It}^{\varepsilon_0}(\mathsf{it}), g, 0) = \vartheta\Omega^{\varepsilon_0}$,

(iii) $|\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA}_0)| \geq \mathsf{it}(g_{q_3})(\omega) = g'_{q_3}(0) = \vartheta\Omega^\Omega$ *(big Veblen number),*

(iv) $|\mathsf{p}_1\mathsf{p}_{n+1}(\mathsf{ACA}_0)| \geq \mathsf{it}(g_{q_3})(\omega) = g'_{q_3}(0) = \vartheta\Omega_n$.

Chapter IV

Notations for the ordinals $(g_x(\alpha) : x \in Q^H)$

In this chapter, we discuss how to obtain notations for the ordinals $(g_x(\alpha) : x \in Q^H)$. Further, we relate the functions $(g_x(\alpha) : x \in Q^H)$ to the ϑ-function as e.g. introduced in Rathjen and Vizcaíno [18], p. 538, for the special case where $|\mathfrak{X}| = \emptyset$.

This chapter makes heavy use of Setzer's work on notations systems [25], in particular his ordinal function generators (OFGs). To keep this thesis reasonably self contained, we have included the proof of Setzer's key lemma (cf. Lemma IV.2.7). After examining how the functions $H_x(f)$ and $H_y(f)$ compare, we use Setzer's OFGs to obtain an ordinal notation system for the ordinal relevant for this work.

Since we build our notation system out of expressions corresponding to $\tilde{g}_x(\alpha)$, a variant of $g_x(\alpha)$, we are left with the task to find notations for the ordinals $g_x(\alpha)$. This is done by defining a recursive function v which is provably total in ACA_0 and assigns to an ordinal expression (x, α) a notation $v(x, \alpha)$ that denotes the ordinal $g_x(\alpha)$.

We conclude the chapter by showing how the functions $(g_x : x \in Q^H)$ relate to the ϑ-function. Together with the main results of Chapter II and Chapter III, this allows to present the ordinals $g_x(\omega)$ of the theories $\mathsf{Op}_x(\mathsf{ACA}_0)$ in a more familiar form.

Convention IV.0.18. *As we work in this chapter exclusively with names $x \in Q^H$, it is assumed that x, y range over Q^H, and we lazily write $x < y$ for $x <^H y$.*

IV.1 How $f_x(\alpha)$ and $f_y(\beta)$ compare

The idea is to represents ordinals by expressions containing only smaller ordinals as their components. Essentially, an ordinal which is not additively principal is written as a sum of smaller additively principal ordinals, and each additively principal ordinal γ is written as $\gamma = g_x(\alpha)$, where α and the ordinals occurring in the name x are below γ, where again, $g(\alpha) := \omega^{1+\alpha}$ is the function which enumerates the infinite principal ordinals.

Definition IV.1.1. *$HZ := \{\omega^\alpha : \alpha \in \mathsf{Ord}\} = \{1, \omega, \omega^2, \ldots\}$ are the additively principal ordinals. We write $\alpha =_{NF} \beta_k + \ldots + \beta_1$ if $\alpha = \beta_k + \ldots + \beta_1$, $\vec{\beta} \in HZ$ and $\beta_1 \leq \ldots \leq \beta_k < \alpha$. Further, $E := \{\gamma : \omega^\gamma = \gamma\}$ is the set of ε-numbers.*

The following is readily observed.

Lemma IV.1.2. *If $1 < \alpha \notin HZ$, then there are unique ordinals $\vec{\beta} \in HZ$, so that $\alpha =_{NF} \beta_k + \cdots + \beta_1$. Further, $\gamma \in E$ iff for all $\alpha, \vec{\beta}$, $\alpha =_{NF} \beta_k + \ldots + \beta_1$ and $\vec{\beta} < \gamma$ implies $\alpha < \gamma$.*

Note that $g = \mathrm{HZ} \setminus \{1\}$, and that $g' = E$. Below, we recall the definition of the components of a name (Definition III.5.5).

Definition IV.1.3. *$k(q_0) := \emptyset$, if $x = \langle(\alpha, y)\rangle$, then $k(x) := \{\alpha\} \cup k(y)$, and $k(\langle x_1, \ldots, x_n \rangle) = k(x_1) \cup \ldots \cup k(x_n)$. And $|x| := \mathsf{max}(k(x))$, where $\mathsf{max}(\emptyset) := -1$.*

Next, we extend $(Q^H, <)$ to $(Q^H \times \Omega, <')$, and the functions $k(\cdot)$ and $|\cdot|$ from Q^H to $Q^H \times \Omega$.

Definition IV.1.4. *For $(x, \alpha), (y, \beta) \in Q^H \times \Omega$, we define*

$$(x, \alpha) <' (y, \beta) :\Leftrightarrow x < y \vee (x = y \wedge \alpha < \beta).$$

Further, $k(x, \alpha) := k(x) \cup \{\alpha\}$, and $|(x, \alpha)| = \mathsf{max}\{k(x, \alpha)\}$.

For a principle ordinal γ there are in general many pairs $(x, \alpha) \in Q^H \times \Omega$ with $|(x, \alpha)| < \gamma$ and $g_x(\alpha) = \gamma$. Thus, we need a way to pick one. To do so, we define some notions which are relative to some $f \in \Omega^{(0)}$. Later, g will take the role of f.

In order to pick a pair (x, α) so that $\gamma = f_x(\alpha)$, we make use of the following properties of the functions $(f_y : y \in Q^H)$.

Lemma IV.1.5. *Let $x \in Q^H$ and $f \in \Omega^{(0)}$.*

(i) If $\deg(x) > 1$ and $o(x) = 1$, then $f_x(\alpha) = f_{x[1+\alpha]}(0)$.

(ii) If $x = y+1$, then $f_x(0) = f_y(\gamma)$ for $\gamma := f_y(0)$, $f_x(\gamma+1) = f_y(\gamma'+1)$, for $\gamma' := f_x(\gamma)$, and if α is a successor, then $f_x(\alpha+1) = f_y(\gamma)$, for $\gamma := f_x(\alpha)$.

Proof (i) is by Lemma III.5.2. (ii) Let $x = y+1$. The first claim is by definition. For the second and third claim, note that $f_x(\gamma) \in (\mathsf{sh} \circ f_y) = f'_y$ (cf. Lemma I.3.15), and that $f_x(\alpha+1) \notin f'(y)$. Thus, the claims are by definition of $f_x = \mathsf{it}(f_y)$. $\square$

In the above lemma, if (i) applies, we regard moving form $f_x(\alpha)$ to $f_{x[1+\alpha]}(0)$ as performing one step in the computation of $f_x(\alpha)$. And similarly, in case (ii), moving from $f_x(0)$ to $f_y(\gamma)$, from $f_x(\gamma+1)$ to $f_y(\gamma'+1)$, or from $f_x(\alpha+1)$ to $f_y(\gamma)$. We can compute in this way, until we arrive that $f_y(\beta)$, where either $y = q_0$ or β is a limit or $o(y)$ is a limit. Such a pair (y, β) where no computation step applies is called good.

Definition IV.1.6. *We say that $(x, \alpha) \in Q^H \times \Omega$ is good, if*

$$x = q_0 \vee o(x) \in \mathsf{Lim}(\Omega) \vee (x = y+1 \wedge \alpha \in \mathsf{Lim}(\Omega)).$$

If (x, α) is not good, then $x = y+1 \wedge \alpha \notin \mathsf{Lim}(\Omega)$, or $\deg(x) > 1 \wedge o(x) = 1$. In these cases, we can perform one of the computation steps shown in Lemma IV.1.5. Since $(Q^H, <)$ is a well-ordering and $x[\alpha] < x$ and $y < y+1$, a good pair is finally reached.

Lemma IV.1.7. *If $(x, \alpha) \in Q^H \times \Omega$ is not good, then there is an (y, β) that is good, so that $\gamma := f_x(\alpha) = f_y(\beta)$, $y < x$, and if $|(x, \alpha)| < \gamma$, then also $|(y, \beta)| < \gamma$.*

Next, we give a first criteria to decide whether for good $(x, \alpha), (y, \beta) \in Q^H \times \Omega$, $f_x(\alpha) \leq f_y(\beta)$.

Lemma IV.1.8. *Assume that (x, α) and (y, β) are good, and that $(x, \alpha) <' (y, \beta)$ and $|(x, \alpha)| < f_y(\beta)$. Then $f_x(\alpha) < f_y(\beta)$.*

Proof Assume that (x, α) and (y, β) are good, and that $(x, \alpha) <' (y, \beta)$ and $|(x, \alpha)| < f_y(\beta)$. If $x = y$, then $\alpha < \beta$ and $f_x(\alpha) = f_y(\alpha) < f_y(\beta)$ as f_y is normal. Hence we may assume that $x < y$. As (y, β) is good, y is either a successor, or $o(y) \in \mathsf{Lim}(\Omega)$. If $y = z+1$ for some z, then β is a limit, and $f_y(\beta) \in f'_z$ by Lemma I.3.15. As $x \leq z$ and $|x| < f_y(\beta)$, we have $f_y(\beta) \in f'_x$ by Lemma III.5.7. Since $\alpha < f_y(\beta)$ by assumption, we obtain $f_x(\alpha) < f_x(f_y(\beta)) = f_y(\beta)$. And if $o(y) = \gamma$, then, as $x < y$, there is a $\xi < \gamma$ so that $x < y[\xi]$ (cf. Lemma III.4.15 (iii)). By Lemma III.5.3 (iii), $f_y(\beta) \in f_y \subseteq f'_{y[\xi]}$. As $x < y$, Lemma III.5.7 yields $f_y(\beta) \in f'_x$. Since $\alpha < f_y(\beta)$ by assumption, $f_x(\alpha) < f_x(f_y(\beta)) = f_y(\beta)$. $\square$

If $x, z \in Q^H$ and $\deg(z) > 1$ and $o(z) = 1$, then the functions f_x and $\alpha \mapsto f_{x[\alpha]}(0)$ have fixed points, hence it may happen that $f_x(\alpha) \in k(x, \alpha)$. Obviously, $f_x(\alpha)$ can possibly only match $|(x, \alpha)|$.

Lemma IV.1.9. *If (x, α) is good and $f_x(\alpha) \in k(x, \alpha)$, then either*

$$f_x(\alpha) = \alpha \in \mathsf{Lim}(\Omega) \qquad \text{or} \qquad f_x(\alpha) = o(x) \in \mathsf{Lim}(\Omega) \wedge \alpha = 0.$$

Proof Assume that (x, α) is good and $f_x(\alpha) \in k(x, \alpha)$. In case that $f_x(\alpha) > \alpha$, then $f_x(\alpha) \in k(x, 0)$, hence $\alpha = 0$ by Lemma III.5.6. As $(x, 0)$ is good by assumption, $o(x)$ is a limit. As for each $\xi < o(x)$, $|x[\xi]| \leq f_{x[\xi]} < f_x(0)$, $|x| = f_x(0)$ is only possible if $f_x(0) = o(x)$. □

Next, we define a finite set $l(x, \alpha) \subseteq k(x, \alpha)$, where $l(x, \alpha)$ is $k(x, \alpha)$ with potential fixed points removed. Note that if e.g. for $x := (2, (\gamma, q_0))$, we have that $x[0] = (1, q_0) * (\gamma, q_0)$, and $x[0]$ contains the additional component 1. To have $l(x) \subseteq k(x)$, we let $l(x) := k(x) \cap k(x[0])$ in this case.

Definition IV.1.10. *To each $(x, \alpha) \in Q^H \times \Omega$, we assign a finite set $l(x, \alpha)$ of ordinals as follows:*

$$l(x, \alpha) := \begin{cases} k(x) & : \alpha \text{ is a limit,} \\ k(x, \alpha) & : \alpha \text{ is a successor, or } \alpha = 0 \wedge o(x) \notin \mathsf{Lim}(\Omega), \\ k(x) \cap k(x[0]) & : \alpha = 0 \wedge o(x) \in \mathsf{Lim}(\Omega). \end{cases}$$

The following is immediate by this definition and Lemma IV.1.9.

Lemma IV.1.11. *If $\beta \in l(x, \alpha)$, then $\beta < f_x(\alpha)$.*

Now we give the complete picture of how $f_x(\alpha)$ and $f_y(\beta)$ compare for good pairs (x, α) and (y, β).

Lemma IV.1.12. *Let $(x, \alpha), (y, \beta) \in Q^H \times \Omega$ be good. Then*

$$\begin{aligned} f_x(\alpha) < f_y(\beta) \quad &\Leftrightarrow \quad ((x, \alpha) <' (y, \beta) \wedge |(x, \alpha)| < f_y(\beta)) \vee f_x(\alpha) < |(y, \beta)| \vee \\ &\qquad f_x(\alpha) \in l(y, \beta) \\ f_x(\alpha) = f_y(\beta) \quad &\Leftrightarrow \quad ((x, \alpha) <' (y, \beta) \wedge f_y(\beta) = |(x, \alpha)| \wedge f_y(\beta) \notin l(x, \alpha)) \vee \\ &\qquad ((y, \beta) <' (x, \alpha) \wedge f_x(\alpha) = |(y, \beta)| \wedge f_x(\alpha) \notin l(y, \beta)) \vee \\ &\qquad (x, \alpha) = (y, \beta) \end{aligned}$$

Proof First formula, right-to-left. The first disjunct on the right implies the left side by Lemma IV.1.8. The other two conjuncts imply the left side due to Lemma III.5.6.

Second formula, right-to-left. We just show that the first disjunct on the right implies the left hand side, as the argument for the second disjunct is symmetric, and trivial for the third. So assume $(x, \alpha) <' (y, \beta) \wedge f_y(\beta) = |(x, \alpha)| \wedge f_y(\beta) \notin l(x, \alpha)$. By Lemma III.5.6, we have $f_y(\beta) = |(x, \alpha)| \leq f_x(\alpha)$. Now we assume $f_y(\beta) < f_x(\alpha)$ and argue for a contradiction. First note that $x = y$ is impossible: then $(x, \alpha) <' (y, \beta)$ yields $\alpha < \beta$, contradicting $f_y(\beta) < f_x(\alpha)$. Hence $x <' y$. By Lemma IV.1.9, we have the following options.

(i) $f_y(\beta) = |(x,\alpha)| = \alpha$ (so α is a limit). Then, $f_y(\beta) \notin l(x,\alpha) = k(x)$, so $|x| < f_y(\beta)$. Hence, for each $\xi < \alpha$, $(x,\xi) <' (y,\beta)$ and $|(x,\xi)| < f_y(\beta)$, therefore, by Lemma IV.1.8, $f_x(\xi) < f_y(\beta)$. As f_x is normal, also $f_x(\alpha) \leq f_y(\beta)$, contradicting $f_y(\beta) < f_x(\alpha)$!

(ii) $f_y(\beta) = |(x,\alpha)| = o(x) =: \gamma$. Then $f_y(\beta) \notin l(x,\alpha)$ implies that $\alpha = 0$. Note that $\mathsf{sup}_{\xi<\gamma} f_{x[1+\xi]}(0) = f_x(0)$. For each $\xi < \gamma$, we have $(x[\xi],0) <' (y,\beta)$ and $|(x[\xi],0)| < f_y(\beta)$, therefore, by Lemma IV.1.8, $f_{x[\xi]} < f_y(\beta)$. Hence, $f_x(0) = f_x(\alpha) \leq f_y(\beta)$, contradicting $f_y(\beta) < f_x(\alpha)$!

First formula, left-to-right. The negation of the right hand side readily implies the conjunction of the following formulas:

(i) $(y,\beta) <' (x,\alpha) \vee (y,\beta) = (x,\alpha) \vee |(x,\alpha)| \geq f_y(\beta)$,

(ii) $f_x(\alpha) > |(y,\beta)| \vee f_x(\alpha) = |(y,\beta)|$,

(iii) $f_x(\alpha) \notin l(y,\beta)$.

We show that the conjunction of (i)–(iii) implies $f_y(\alpha) \leq f_x(\beta)$. This follows since $(y,\beta) = (x,\alpha) \vee |(x,\alpha)| \geq f_y(\beta)$ implies $f_y(\alpha) \leq f_x(\beta)$ (cf. Lemma III.5.6), and $(y,\beta) <' (x,\alpha) \wedge f_x(\alpha) > |(y,\beta)|$ implies $f_y(\alpha) \leq f_x(\beta)$ (Lemma IV.1.8), and furthermore, $(y,\beta) <' (x,\alpha) \wedge f_x(\alpha) = |(y,\beta)| \wedge f_x(\alpha) \notin l(y,\beta)$ implies $f_x(\alpha) = f_y(\beta)$ (second formula, right-to-left).
Second formula, left-to-right. The negation of the righthand side readily implies the conjunction of the following formulas:

(i) $(y,\beta) <' (x,\alpha) \vee f_y(\beta) > |(x,\alpha)| \vee f_y(\beta) < |(x,\alpha)| \vee f_y(\beta) \in l(x,\alpha)$,

(ii) $(x,\alpha) <' (y,\beta) \vee f_x(\alpha) > |(y,\beta)| \vee f_x(\alpha) < |(y,\beta)| \vee f_x(\alpha) \in l(y,\beta)$,

(iii) $(x,\alpha) \neq (y,\beta)$.

We show that the conjunction of (i)–(iii) implies $f_x(\alpha) \neq f_y(\beta)$. This follows, since $(y,\beta) <' (x,\alpha) \wedge (x,\alpha) <' (y,\beta) \wedge (x,\alpha) \neq (y,\beta)$ is impossible, and $f_x(\alpha) \neq f_y(\beta)$ follows if $f_y(\beta) \in l(x,\alpha)$, or if $f_x(\alpha) \in l(y,\beta)$ (cf. Lemma IV.1.11), or if $f_y(\beta) < |(x,\alpha)|$, or if $f_x(\alpha) < |(y,\beta)|$ (cf. Lemma III.5.6), or if $(x,\alpha) \neq (y,\beta) \vee f_y(\beta) > |(x,\alpha)| \vee f_y(\beta) > |(x,\alpha)|$ (cf.Lemma IV.1.8). □

This characterizations becomes considerably simpler if we impose the additional restrictions that $|(x,\alpha)| < f_x(\alpha)$ and $|(y,\beta)| < f_y(\beta)$.

Corollary IV.1.13. *Let $(x,\alpha),(y,\beta)$ be good, so that $|(x,\alpha)| < f_x(\alpha)$ and $|(y,\beta)| < f_y(\beta)$. Then, $f_x(\alpha) = f_y(\beta) \Leftrightarrow (x,\alpha) = (y,\beta)$, and $f_x(\alpha) < f_y(\beta)$ iff either*

(i) $(x,\alpha) <' (y,\beta) \wedge |(x,\alpha)| < f_y(\beta)$, *or*

(ii) $(y, \beta) \leq' (x, \alpha) \wedge f_x(\alpha) \leq |(y, \beta)|$.

Proof Just observe that by the extra assumptions $|(x, \alpha)| < f_x(\alpha)$ and $|(y, \beta)| < f_y(\beta)$, the possibilities $f_y(\beta) < |(x, \alpha)|$, $f_y(\beta) = |(x, \alpha)|$ and $f_x(\alpha) = |(y, \beta)|$ are ruled out. So the claim follows readily by the above lemma. $\square$

The restriction to good pairs (x, α) is a bit awkward. Note that only if x is of the form $y{+}1$ does it depend on α whether (x, α) is good. To improve the situation, we arrange things so that each pair $(x, \alpha) \in Q^H \times \Omega$ that is good corresponds to a pair $(y, \beta) := \mathsf{good}(x, \alpha) \in \tilde{Q}_0 \times \Omega$ so that $f_x(\alpha) = \tilde{f}_y(\beta)$.

Definition IV.1.14. *We define $\tilde{Q}_0$, $\tilde{Q}$ and $\mathsf{good} : \tilde{Q}_0 \times \Omega \to Q \times \Omega$ as follows.*

(i) $\tilde{Q}_0 := \{x \in Q^H : \deg(x) > 1 \to o(x) \in \mathsf{Lim}(\Omega)\}$ *and* $\tilde{Q} := \tilde{Q}_0 \setminus \{q_0\}$.

(ii)

$$\mathsf{good}(x, \alpha) := \begin{cases} (x, \alpha) & : x = q_0 \vee o(x) \in \mathsf{Lim}(\Omega), \\ (x, \omega(1{+}\alpha)) & : \deg(x) = 1 \wedge o(x) = 1. \end{cases}$$

Definition IV.1.15. *If $x \in \tilde{Q}_0$, then*

$$\tilde{f}_x := \begin{cases} \mathsf{sh} \circ f_x & : \deg(x) = 1 \wedge o(x) = 1, \\ f_x & : \textit{else}. \end{cases}$$

Observe that $\tilde{f}_{y+1} := f_y$ (cf. Lemma I.3.15), and if $x = q_0 \vee o(x) \in \mathsf{Lim}(\Omega)$, then $\tilde{f}_x := f_x = \bigcap_{\xi<\gamma} f'_{x[\xi]}$ (cf. Lemma III.5.3). In particular, if $x > q_0$, then $\tilde{f}_x \subseteq f'$.

Further, as f_x and $\tilde{f}_x$ have the same fixed points (cf. Lemma I.3.15), and further $|y| \trianglelefteq f_y(0) \triangleleft f_{y+1}(0) \in \mathsf{Lim}(\Omega)$ and so $|y{+}1| \triangleleft f_{y+1}(0)$, we again have the following.

Lemma IV.1.16. *If $\tilde{f}_x(\alpha) \in k(x, \alpha)$, then $\tilde{f}_x(\alpha) = \alpha$ or $\tilde{f}_x(\alpha) = o(x) \wedge \alpha = 0$.*

Also the following is readily observed.

Lemma IV.1.17.

(i) $\mathsf{good}[\tilde{Q}_0 \times \Omega] = \{(x, \alpha) \in Q^H \times \Omega : (x, \alpha)$ *is good*$\}$.

(ii) $\mathsf{good} : (\tilde{Q}_0 \times \Omega, <') \to (\mathsf{good}[\tilde{Q}_0 \times \Omega], <')$ *is an order isomorphism.*

(iii) *If $(x, \alpha) \in \tilde{Q}_0 \times \Omega$ and $(y, \beta) := \mathsf{good}(x, \alpha)$, then $\tilde{f}_x(\alpha) = f_y(\beta)$.*

If $\gamma := \tilde{f}_y(\beta)$, then either $y = z{+}1$ and $\tilde{f}_y = f'_y$, or $o(y) = \gamma'$ for some limit γ', and $\tilde{f}_y = f_{y[\gamma']}x = \bigcap_{\xi<\gamma'} f'_{y[\xi]} \subseteq f'_y[0]$ (cf. Lemma III.5.3). Therefore, by Remark I.3.8, $\gamma = \omega(1{+}\beta)$ for some β, and thus $\alpha \lhd \gamma$ iff $\omega\alpha \lhd \gamma$. Further, $l(y{+}1,\gamma) = k(y{+}1) = l(y{+}1, \omega(1{+}\gamma))$. The lemma below summarizes this discussion.

Lemma IV.1.18. *Let $(x,\alpha),(y,\beta) \in \tilde{Q}_0 \times \Omega$, $(x'\alpha') := \mathsf{good}(x,\alpha)$ and $\gamma = \tilde{f}_y(\beta)$. Then, $|(x,\alpha)| \lhd \gamma$ iff $|(x',\alpha')| \lhd \gamma$, and $\gamma \in l(x,\alpha)$ iff $\gamma \in l(x',\alpha')$.*

As a result, we have the following.

Lemma IV.1.19. *For all $(x,\alpha),(y,\beta) \in \tilde{Q}_0 \times \Omega$ we have the following.*

$$\begin{aligned}
\tilde{f}_x(\alpha) < \tilde{f}_y(\beta) \quad &\Leftrightarrow \quad ((x,\alpha) <' (y,\beta) \wedge |(x,\alpha)| < \tilde{f}_y(\beta)) \vee \tilde{f}_x(\alpha) < |(y,\beta)| \vee \\
&\qquad \tilde{f}_x(\alpha) \in l(y,\beta) \\
\tilde{f}_x(\alpha) = \tilde{f}_y(\beta) \quad &\Leftrightarrow \quad ((x,\alpha) <' (y,\beta) \wedge \tilde{f}_y(\beta) = |(x,\alpha)| \wedge \tilde{f}_y(\beta) \notin l(x,\alpha)) \vee \\
&\qquad ((y,\beta) <' (x,\alpha) \wedge \tilde{f}_x(\alpha) = |(y,\beta)| \wedge \tilde{f}_x(\alpha) \notin l(y,\beta)) \vee \\
&\qquad (x,\alpha) = (y,\beta)
\end{aligned}$$

IV.2 Ordinal function generators (OFGs)

In this section, we look at ordinal expressions and ordinal function generators as introduced in Setzer [25]. These tools are then used to obtain ordinal notations, and to relate the functions $(g_x : x \in \tilde{Q})$ to the ϑ-function. Before we review Definition and Lemma 2.5 from Setzer [25], we start with a specific instance of what is later called an ordinal function generator (OFG).

Below, we define a set of ordinal expressions A and a well-ordering $<'$ on A. Then, we assign to each $a \in A$ an ordinal $G(a)$.

Definition IV.2.1. *The set A is defined as follows.*

(i) $\Sigma_{asc} := \{0\} \cup \{\langle \alpha_0, \ldots, \alpha_k \rangle : \alpha_0 \le \ldots \le \alpha_k\}$ *(we identify 0 with $\langle\rangle$).*

(ii) $A := \Sigma_{asc} \cup \tilde{Q} \times \Omega$.

The intended value of an ordinal expression $a \in A$ is given by $G(a)$ defined below.

Definition IV.2.2. *We define $G : A \to \Omega$ as follows.*

(i) $G(0) := 0$,

(ii) $G(\langle \alpha_0, \ldots, \alpha_k \rangle) := \omega^{\alpha_k} + \ldots + \omega^{\alpha_0}$,

(iii) $G(x,\alpha) := \tilde{g}_x(\alpha)$.

Definition IV.2.3. *We extend the well-ordering* $(Q^H, <)$ *to* $(A, <')$ *as follows.*

(i) If $a \in \Sigma_{asc}$ *and* $b \in \tilde{Q} \times \Omega$, *then* $a <' b$,

(ii) $\langle \alpha_0, \ldots, \alpha_k \rangle <' \langle \beta_0, \ldots, \beta_l \rangle$ *iff* $\alpha_k < \beta_l$ *or there is an* $i < \mathsf{min}\{k, l\}$ *so that* $\langle a_{k-i}, \ldots, a_k \rangle = \langle b_{l-i}, \ldots, b_l \rangle$ *and* $a_{k-i-1} < b_{l-i-1}$.

(iii) $(x, \alpha) <' (b, \beta)$ *iff* $x < y \vee (x = y \wedge \alpha < \beta)$.

We point out that for $a, b \in \Sigma_{asc}$, $a <' b$ iff $G(a) < G(b)$.

Next, we extend $|\cdot|$, k and l from $Q^H \times \Omega$ to A by saying what these functions do with elements in Σ_{asc}.

Definition IV.2.4. *If* $\sigma := \langle \alpha_0, \ldots, \alpha_k \rangle$, *then* $k(\sigma) := \{\alpha_0, \ldots, \alpha_k\}$. *In case that* α_0 *is a limit and* $\mathsf{lh}(\sigma) = 1$, *then* $l(\sigma) = \emptyset$, *and if* α_0 *is not a limit or* $\mathsf{lh}(\sigma) > 1$, *then* $l(\sigma) := k(\sigma)$. *Further,* $k(0) = l(0) = \emptyset$, *and* $|\sigma| := \mathsf{max}(k(\sigma))$ *(where* $\mathsf{max}(\emptyset) := -1$*).*

Note that $k(a)$ is a finite set of ordinals. Further, if $k(a) \neq \emptyset$, then $\bigcup k(a) = |a|$, and as $\alpha+1 = \alpha \cup \{\alpha\}$, $|a| \cup k(a) = |a| \cup \{|a|\} = |a|+1$, and moreover, since $l(a) \subseteq k(a)$, we have that $|a| \cup l(a) = |a|$ if $|a| \notin l(a)$, and $|a| \cup l(a) = |a|+1$ if $|a| \in l(a)$. That all this also holds for $k(a) = \emptyset$, we let for each $X \subseteq \mathsf{Ord}$, $-1 \cup X := X$, and we consider $-1 \subseteq X$ as true, and $-1 \in X$ as false.

The quadruple $\mathcal{O}_H := (A, <', k, l)$ is an ordinal function generator in the sense of the next definition. Definitions IV.2.5 and IV.2.8, and Lemma IV.2.7 correspond to Lemma 2.5 in Setzer [25]. As there, if f is a function and $X \subseteq \mathsf{dom}(f)$, then $f[X] := \{f(x) : x \in X\}$. Further, $X \subseteq_{\mathsf{fin}} Y$ states that X is a finite subset of Y.

Definition IV.2.5. *A quadruple* $\mathcal{O} := (Arg, <', k, l)$ *consisting of a well-ordering* $(Arg, <')$ *and functions* k, l *so that for each* $a \in Arg$, $l(a) \subseteq k(a) \subseteq_{\mathsf{fin}} \mathsf{Ord}$, *is called an ordinal function generator (OFG). By recursion on* $(Arg, <')$ *we simultaneously define for each* $a \in Arg$ *a set* $C(a) \subseteq \mathsf{Ord}$, *and a function* $eval : Arg \to \mathsf{Ord}$. *Further, we define sets* NF, Cl *and* $Arg[Cl] \subseteq Arg$. *If we want to indicate that the objects* $eval, C, NF, Cl$ *and* $Arg[Cl]$ *are induces by the OFG* $\mathcal{O}$, *we denote them by* $eval_{\mathcal{O}}, C_{\mathcal{O}}, NF_{\mathcal{O}}, Cl_{\mathcal{O}}$ *and* $Arg[Cl_{\mathcal{O}}]$, *respectively.*

(i) $C^0(a) = |a| \cup l(a)$, $C^{n+1}(a) = \{eval(b) : b <' a \wedge k(b) \subseteq C^n(a)\}$ *and* $C(a) := \bigcup_{n<\omega} C^n(a)$. *Further,* $eval(a) := \mathsf{min}\{\alpha : \alpha \notin C(a)\}$.

(ii) $NF := \{a \in Arg : |a| < eval(a)\}$.

(iii) $Cl^0 := \emptyset$, $Cl^{n+1} := \{a \in NF : k(a) \subseteq eval[Cl^n]\}$ *and* $Cl := \bigcup_{n<\omega} Cl^n$.

(iv) $Arg[Cl] := \{a \in Arg : k(a) \subseteq eval[Cl]\}$.

Since $|a| \subseteq C(a)$, $\text{eval}(a) \geq |a|$. So if $b \notin \text{NF}$, then $\text{eval}(b) = |b| \in k(b)$. Further, as $k(a)$ is finite, $C(a)$ is the least set X so that

$$|a| \cup l(a) \subseteq X \quad \text{and} \quad \{\text{eval}(b) : b \in \text{NF} \wedge b <' a \wedge k(b) \subseteq X\} \subseteq X$$

(if $b \notin \text{NF} \wedge b <' a \wedge k(b) \subseteq X$, then already $\text{eval}(b) = |b| \in k(b) \subseteq X$). Also by the finiteness of $k(a)$, $\text{Cl} \subseteq \text{NF}$ is the least set Y so that $\{a \in \text{NF} : k(a) \subseteq \text{eval}[Y]\} \subseteq Y$.

Lemma IV.2.6. *If $C^0(b) \subseteq C(a)$ and $b <' a$, then $C(b) \subseteq C(a)$.*

Proof Assume that $C^0(b) \subseteq C(a) \wedge b <' a$. We show $(\forall n \in \mathbb{N}) C^n(b) \subseteq C(a)$ by induction on n. for $n = 0$ there is nothing to show, and if $\text{eval}(c) \in C^{n+1}(b) \setminus C^n(b)$, then $c <' b <' a$ and $k(c) \subseteq C^n(b) \subseteq_{I.H.} C(a)$, hence $\text{eval}(c) \in C(a)$. □

Lemma IV.2.7.

(i) $C(a)$ is an initial segment of the ordinals.

(ii) $eval{\restriction}NF$ is injective.

(iii) $eval[Cl]$ is an initial segment of the ordinals.

(iv) For $a, b \in NF$ we have

$$eval(a) < eval(b) \quad \Leftrightarrow \quad (a <' b \wedge |a| < eval(b)) \vee (b \leq' a \wedge eval(a) \leq |b|).$$

(v) For $a, b \in Arg[Cl]$,

$$\begin{aligned}
eval(a) < eval(b) \quad &\Leftrightarrow \quad (a <' b \wedge |a| < eval(b)) \vee eval(a) < |b| \vee eval(a) \in l(b) \\
eval(a) = eval(b) \quad &\Leftrightarrow \quad (a <' b \wedge eval(b) = |a| \wedge eval(b) \notin l(a)) \vee \\
&\qquad (b <' a \wedge eval(a) = |b| \wedge eval(a) \notin l(b)) \vee \\
&\qquad a = b
\end{aligned}$$

Proof (i) To prove that $C(a)$ is an ordinal, we show by induction on a w.r.t. $<'$ and side induction on n that $(\forall \alpha \in C^n(a))(\alpha \subseteq C(a))$. Note that if $C(a)$ is an ordinal, then $\text{eval}(a) = C(a)$. $C^0(a) = |a| \cup l(a)$ is either $|a|$ or $|a|+1$, hence an ordinal. Thus, if $\alpha \in C^0(a)$, then $\alpha \subseteq C^0(a) \subseteq C(a)$. If $\text{eval}(b) \in C^{n+1}(a) \setminus C^n(a)$, then $b <' a$ and $k(b) \subseteq C^n(a)$, so $|b| \in C^n(a)$ and $l(b) \subseteq C^n(a)$. By side I.H., $|b| \subseteq C(a)$, thus $|b| \cup l(b) = C^0(b) \subseteq C(a)$, so by Lemma IV.2.6, $C(b) \subseteq C(a)$. As $b <' a$, the main I.H. yields that $C(b)$ is an ordinal. Therefore $\text{eval}(b) = C(b) \subseteq C(a)$.

(ii) Assume that $a, b \in \text{NF}$, $b <' a$ and $\text{eval}(a) = \text{eval}(b)$. We have $|b| < \text{eval}(b) = \text{eval}(a) =_{(i)} C(a)$, so $k(b) \subseteq C(a)$. Since $b <' a$, $\text{eval}(b) \in C(a) = \text{eval}(a)$, contradicting $\text{eval}(a) = \text{eval}(b)$!

(iii) We show by main induction on a w.r.t. $<'$ and side induction on n that $C^n(a) \subseteq \text{eval}[\text{Cl}]$. This implies the claim: if $a \in \text{Cl}$ and $\beta < \text{eval}(a)$, then for some n, $\beta \in C^n(a) \subseteq \text{eval}[\text{Cl}]$. Fix $a \in \text{Cl}$. By definition of Cl, $l(a) \subseteq k(a) \subseteq \text{eval}(a)$, in particular, $|a| \in \text{eval}[\text{Cl}]$. Thus, $|a| = \text{eval}(b) \in \text{eval}(a)$ for some $b \in \text{Cl}$. If $a <' b$, then $k(a) \subseteq C(b)$ would entail $\text{eval}(a) \in C(b)$! Hence $b <' a$, and by I.H., $\text{eval}(b) = C(b) \subseteq \text{eval}[\text{Cl}]$. That is, $C^0(a) = |a| \cup l(a) \subseteq \text{eval}[\text{Cl}]$. And if $\text{eval}(c) \in C^{n+1}(a) \setminus C^n(a)$ for some c, then $c <' a$ and $k(c) \subseteq C^n(a) \subseteq_{s.I.H.} \text{eval}[\text{Cl}]$. As $\text{eval}(c) \neq |c|$, $c \in \text{Cl}$ follows. Hence $C^{n+1}(a) \subseteq \text{eval}[\text{Cl}]$.

(iv) Assume that $a, b \in \text{NF}$. Left-to-right. Assume $\text{eval}(a) < \text{eval}(b)$. Then $|a| < \text{eval}(b)$. If $a <' b$ we are done, and if $b \leq' a$, then also $b <' a$. If we had $|b| < \text{eval}(a)$, then $b \in C(a)$, contradicting $\text{eval}(a) < \text{eval}(b)$, so $\text{eval}(a) \leq |b|$.
Right-to-left. $a <' b$ and $|a| < \text{eval}(b)$ implies $\text{eval}(a) \in C(b) = \text{eval}(b)$, therefore $\text{eval}(a) < \text{eval}(b)$. And if $\text{eval}(a) \leq |b|$, the claim follows from $|b| < \text{eval}(b)$.

(v) Using (iv) and proceeding as in the proof of Lemma IV.1.12. □

We like to point out that (v) becomes much simpler if we have that $l(a) = k(a)$. In this case, $|a| < \text{eval}(a)$, $\alpha \in l(a)$ implies $\alpha \leq |a|$, and $\alpha = |a|$ implies $\alpha \in l(a)$. Hence, if $a, b \in \text{Arg}[\text{Cl}] \wedge l(a) = k(a) \wedge l(b) = k(b)$, then $\text{eval}(a) = \text{eval}(b) \Leftrightarrow a = b$, and

$$\text{eval}(a) < \text{eval}(b) \Leftrightarrow (a <' b \wedge |a| < \text{eval}(b)) \vee \text{eval}(a) \leq |b|.$$

Knowing that $\text{eval}{\restriction}\text{Cl}$ is injective, we can assign to each $a \in \text{Arg}[\text{Cl}]$ a finite set $k^0(a) \subseteq \text{Cl}$ as done below, so that $\text{eval}[k^0(a)] = k(a)$. Further, $\text{length}(a)$ assigns to each $a \in \text{Arg}[\text{Cl}]$ a length that is bigger than the length of each $b \in k^0(a)$.

Definition IV.2.8. *$k^0(a) := eval^{-1}[k(a)] \cap Cl$ and $l^0(a) := eval^{-1}[l(a)] \cap Cl$. Further, $length(0) := 0$, and if $0 \neq a \in Arg[Cl]$, then $length(a) = \mathsf{max}(length[k^0(a)])+1$.*

IV.2.1 The OFG $\mathcal{O}_H := (A, <', k, l)$

Now we look at the OFG $\mathcal{O}_H := (A, <', k, l)$ with $A = \Sigma_{asc} \cup \tilde{Q} \times \Omega$, and k and l as specified by Definition IV.2.4. Below, NF, Cl and eval are w.r.t. the OFG $\mathcal{O}_H$. We aim to show that for each $a \in A$, $\text{eval}(a) = G(a)$.

By definition, $\text{eval}(a)$ is the least ordinal not in $C(a)$, that is, if $a = (x, \alpha)$, then by definition of $C(a)$ and since $C(a)$ is an ordinal, $\text{eval}(a)$ is the least ordinal ξ so that $|a| \cup l(a) \subseteq \xi$ and for each $b <' a$ with $|b| < \xi$, also $\text{eval}(b) < \xi$. Exactly the same holds for $G(a)$.

Lemma IV.2.9. *For each $A \ni a := (x, \alpha) \in \tilde{Q} \times \Omega$, we have that $G(x, \alpha) = \xi_0$, where*

$$\xi_0 := \mathsf{min}\{\xi \supseteq |a| \cup l(a) : (\forall b <' a)(|b| < \xi \Rightarrow G(b) < \xi)\}.$$

Proof By Lemmas III.5.6 and IV.1.19, we have that $G(x, \alpha) \supseteq |a| \cup l(a)$ and that $b <' a \wedge |b| < G(x, \alpha)$ implies $G(b) < G(x, \alpha)$, hence $\xi_0 \leq G(x, \alpha)$. Next, we show that $G(x, \alpha) \leq \xi_0$. If $(x, \alpha) = (q_1, 0)$, $G(q_1, 0) = \tilde{g}_{q_1}(0) = (\mathsf{sh} \circ \mathsf{it})(g, 0) = g'(0) = \varepsilon_0$, and $\varepsilon_0 \leq \xi_0$ follows, as it is readily seen that $\xi_0 \in E$. If $(x, \alpha) \neq (q_1, 0)$, then we provide a sequence $(b_i \in A : i \in I)$ with $\mathsf{sup}_{i \in I} G(b_i) = G(x, \alpha)$ and for each $i \in I$, $G(b_i) < \xi_0$. We do so by a case distinction on the form of (x, α).

(i) α is a limit. Then $l(x, \alpha) = k(x)$, hence $|x| < \xi_0$ and $\alpha \leq \xi_0$. For each $\beta < \alpha$, let $b_\beta := (x, \beta)$. Then, for each $\alpha < \beta$, $(x, \beta) <' (x, \alpha)$ and $|(x, \beta)| < \xi_0$, so also $G(x, \beta) < \xi_0$. And by Lemma III.5.4 (i), $\mathsf{sup}_{\beta < \alpha} G(x, \beta) = G(x, \alpha)$ (recall that $G(x, \beta)$ is $g_x(\omega(1+\beta))$ if $\deg(x) = 1$ and $o(x) = 1$, and $g_x(\beta)$ otherwise).

(ii) $(x, \alpha) = (y+1, \beta+1)$. Then $l(x, \alpha) = k(x, \alpha)$, $|(x, \alpha)| < \xi_0$, and as $\xi_0 \in E$, $\xi_0 > \beta+\omega$. Further, we have that $G(x, \alpha) = g_x(\gamma_0 + \omega)$ for $\gamma_0 := \omega(1+\beta)$. Let $b_0 := \langle 1, (x, \beta) \rangle$ and $b_{n+1} := (y, G(b_n))$ if $\deg(y) = 1$ and $o(y) = 1$, and $b_{n+1} := (y[G(b_n)], 0)$ otherwise. Note that $G(b_n)$ is a limit and thus $b_n \in \tilde{Q} \times \Omega$. By induction on n, we show that $G(x, \gamma_0+n) \leq G(b_n) < \xi_n$. As $\langle 1, (x, \beta) \rangle <'$ (x, α) and $|\langle 1, (x, \beta) \rangle| < \xi_0$, we have $G(x, \gamma_0) < G(b_0) = G(\langle 1, (x, \beta) \rangle) < \xi_0$. And if $G(b_n) < \xi_0$, then $|b_{n+1}| < \xi_0$, so as $b_{n+1} <' (x, \alpha)$, also $G(b_{n+1}) < \xi_0$. If $G(x, \gamma_0+n) < G(b_n)$, then $G(x, \gamma_0+n+1)$ is either below $G(y, G(b_n))$ or $G(y[G(b_n)], 0)$ by definition of it and since $\mathsf{it}(g_y, \gamma_0) \in g'_y$.

(iii) $o(x) = \gamma$ and $\alpha = 0$. For each $\beta < \gamma$, let $b_\beta := (x[\beta], \omega)$. Then, for each $\beta < \gamma$ $(x[\beta], \omega) <' a$ and $|(x[\beta], \omega)| < \xi_0$, hence $G((x[\beta], \omega)) < \xi_0$. Using Lemma III.5.4 and III.5.3 yields that $G(x, 0) = \mathsf{sup}_{\beta < \gamma} G(b_\beta)$.

(iv) $o(x) = \gamma$ and $\alpha = \beta+1$. Then, $l(x, \alpha) = k(x, \alpha)$ and $|(x, \beta+1)| < \xi_0$. We let $b_0 := \langle 0, (x, \beta) \rangle$, $b_1 := G(x[\xi], G(b_\xi)+\omega)$, if $1 < \xi > \gamma$, then $b_{\xi+1} :=$ $G(x[\xi], G(b_\xi))$, and if $\gamma' < \gamma$, $b_{\gamma'} := G(x[\gamma'], G(b_0))$. Similar as in case (ii), one shows by transfinite induction on ξ using Lemma III.5.4 that for each $\xi < \gamma$, $G(b_\xi) < \xi_0$, and that $\xi_0 = \mathsf{sup}_{\xi < \gamma} G(b_\xi)$.

□

The above holds also for $a \in \Sigma_{asc}$.

Lemma IV.2.10. *For each $a \in \Sigma_{asc}$, we have that $G(a) = \xi_0$, where*

$$\xi_0 := \mathsf{min}\{\xi \supseteq |a| \cup l(a) : (\forall b <' a)(|b| < \xi \Rightarrow G(b) < \xi)\}.$$

Proof Again, $G(x, \alpha) \supseteq |a| \cup l(a)$, and as $a \in \Sigma_{asc}$, $b <' a$ implies that also $b \in \Sigma_{asc}$, and so $G(b) < G(a)$. Hence, $\xi_0 \leq G(a)$. Next, we show that $G(a) \leq \xi_0$. If $a = 0$, the claim is readily checked, and if $G(a) \in E$, then $|a| = G(a) \leq \xi_0$. In case that $G(a) \notin E$, then $a = \langle \alpha_0, \dots, \alpha_k \rangle$. If $\alpha_0 = 0$, then $b := \langle \alpha_1, \dots, \alpha_{k-1} \rangle <' a$ and

$k(b) < \xi_0$, so $G(b) < \xi_0$, thus $G(a) = G(b)+1 \leq \xi_0$. If $\alpha_0 = \beta+1$, then $l(a) = k(a)$, and we have for each $b_n := \langle \beta \dots, \beta, \alpha_1, \dots, \alpha_{k-1} \rangle$ of length $k+n$ that $b <' a$ and $|b| < \xi_0$, hence $G(b_n) < \xi_0$. As $G(a) = \mathsf{sup}_n(G(b_n))$, $G(a) \leq \xi_0$ follows. And if $\alpha_0 = \gamma$, then either $a = \langle \gamma \rangle$, or $\mathsf{lh}(a) > 1$ and $l(a) = k(a)$ and $|a|+1 \leq \xi_0$, hence for each $\beta < \gamma$, $b_\beta := \langle \beta, \alpha_1, \dots, \alpha_k \rangle <' a$ and $|b_\beta| < \xi_0$, thus $G(b_\beta) < \xi_0$. Again, as $G(a) = \mathsf{sup}_{\beta<\gamma} G(b_\beta)$, $G(a) \leq \xi_0$ follows. □

Combining the two previous lemmas yields the following.

Corollary IV.2.11. *For each $a \in A$, we have that $G(a) = \xi_0$, where*

$$\xi_0 := \mathsf{min}\{\xi \supseteq |a| \cup l(a) : (\forall b <' a)(|b| < \xi \Rightarrow G(b) < \xi)\}.$$

The following is now readily obtained by induction on $(A, <')$.

Lemma IV.2.12. *For each $a \in A$, eval$(a) = G(a)$.*

Proof Let $a \in A$ and suppose that the claim holds for each $b <' a$ with $b \in A$. By the above corollary,

$$\text{eval}(a) = \mathsf{min}\{\xi \supseteq |a| \cup l(a) : (\forall b <' a)(|b| < \xi \Rightarrow \text{eval}(b) < \xi)\} =_{IH}$$
$$\mathsf{min}\{\xi \supseteq |a| \cup l(a) : (\forall b <' a)(|b| < \xi \Rightarrow G(b) < \xi)\} = G(a).$$

□

Summing up, we can characterize the ordinal eval[Cl] $= G$[Cl] in the following way.

Corollary IV.2.13.

(i) eval[Cl] $= G$[Cl] is the least ordinal Λ so that for each $(x, \alpha) \in \tilde{Q} \times \Omega$ with $|(x, \alpha)| < \Lambda$, also $\tilde{g}_x(\alpha) < \Lambda$.

(ii) For $a \prec b :\Leftrightarrow$ eval$(a) <$ eval(b), the map eval : $(Cl, \prec) \to ($eval$[Cl], \in)$ is an order-isomorphism.

IV.2.2 Fixed point free variants of the functions $(\tilde{g}_x : x \in \tilde{Q})$

Let $\mathcal{O} = (\text{Arg}, <', k, l)$ be some OFG. In some sense, for $a \in \text{Arg} \setminus \text{NF}$, eval$(a)$ is a fixed point of eval, as eval$(a) \in k(a)$. Note that if $|a| \in l(a)$ for some $a \in \text{Arg}$, then $C^0(a) = |a| \cup l(a) = |a| \cup \{|a|\} = |a|+1$, and therefore $|a| <$ eval(a), that is $a \in \text{NF}$. In particular, if k and l agree on $A_0 \subseteq \text{Arg}$, then $A_0 \subseteq \text{NF}$.

Next, we consider the OFG $\mathcal{O}_{H'} = (A, <', k, l_{H'})$ which is obtained from the OFG $\mathcal{O}_H = (A, <', k, l)$ by changing l to $l_{H'}$ according to the following definition. We examine how eval$_H :=$ eval$_{\mathcal{O}_H}$ and eval$_{H'} :=$ eval$_{\mathcal{O}'_H}$ relate. In order to describe this relationship, we consider the order-isomorphism introduced in the next lemma.

Definition IV.2.14. *Let $\mathcal{O}_H = (A, <', k, l)$ be the OFG from the previous subsection, and*

$$l_{H'}(a) := \begin{cases} k(a) & : a \in \tilde{Q} \times \Omega, \\ l(a) & : a \notin \tilde{Q} \times \Omega. \end{cases}$$

Further, $\mathcal{O}_{H'} := (A, <', k, l_{H'})$, and $eval_{H'} := eval_{\mathcal{O}_{H'}}$, $Cl_{H'} := Cl_{\mathcal{O}_{H'}}$ and $NF_{H'} := NF_{\mathcal{O}_{H'}}$.

Hence $l_{H'}\restriction(\tilde{Q} \times \Omega) = k\restriction(\tilde{Q} \times \Omega)$ and thus $\tilde{Q} \times \Omega \subseteq \mathrm{NF}_{H'}$.

Given a well-ordering $(X, <)$ (so it makes sense to speak about $x{+}1 := x{+}_{<}1$ and $x{+}n$) which is closed under successors, and a set $L \subseteq X$ of elements without an immediate $<$-predecessor (0 or a limit), we consider a canonic order-isomorphism between $(X, <)$ and $(X \setminus L, <)$: to an $x \in L$, we simply assign $x{+}1$, and to obtain an order-isomorphism, an element of the form $x{+}n$ for $x \in L$ is mapped to $x{+}n{+}1$.

Lemma IV.2.15. *Let $(X, <)$ be a well-ordering where X is closed under successor, and $L \subseteq X$ a set of elements without an immediate $<$-predecessor. Then, the map $\oplus_L : (X, <) \to (X \setminus L, <)$, given by*

$$x \oplus_L 1 := \begin{cases} x{+}1 & : (\exists y \in L)(\exists n \in \mathbb{N})(x = y{+}n), \\ x & : else, \end{cases}$$

is an order-isomorphism.

Next, let $L := (\tilde{Q} \times \Omega) \setminus \mathrm{NF}_H$ (so $(\tilde{Q} \times \Omega) \cap \mathrm{NF}_H = (\tilde{Q} \times \Omega) \setminus L$). Note that $(x, \alpha){+}n = (x, \alpha{+}n)$. Hence, for $\delta \in \mathsf{Lim}(\Omega) \cup \{0\}$, $(x, \delta{+}n) \oplus_L 1$ is $(x, \delta{+}n)$ if $(x, \delta) \in \mathrm{NF}$, and otherwise, $(x, \delta{+}n) \oplus_L 1 = (x, \delta{+}n{+}1)$.

Below, list some further simple observation.

Lemma IV.2.16. *Let $\oplus := \oplus_L$ (for $L := (\tilde{Q} \times \Omega) \setminus NF_H$). Then we have the following for each $a \in A = \Sigma_{asc} \cup (\tilde{Q} \times \Omega)$.*

(i) If $\langle\rangle \neq a \in \Sigma_{asc}$, then $a{+}1 \in \Sigma_{asc}$ and $|a| = |a{+}1|$.

(ii) If $a \in \tilde{Q} \times \Omega$, then $a{+}1 \in \tilde{Q} \times \Omega$ and $|a| \leq |a{+}1| \leq |a|{+}1$.

(iii) $a{+}1 \in NF_H$, $a{+}1 \in NF_{H'}$, and $\tilde{Q} \times \Omega \subseteq NF_{H'}$.

(iv) $eval_H(a) < eval_H(a{+}1)$.

(v) If $a \oplus 1 = a$, then $a \in \Sigma_{asc}$, or $a \in \tilde{Q} \times \Omega$ and $a = b{+}n$ for $b = 0$ or some limit $b \in NF_H$.

(vi) If $a \in \tilde{Q} \times \Omega$, then $a \oplus 1 \in NF_H$.

Proof (i) Note that if $a \in \Sigma_{asc}$, then $a{+}1 = \langle 0 \rangle * a$. (ii) As $(x, \alpha){+}1 = (x, \alpha{+}1)$. (iii) $l(a{+}1) = k(a{+}1)$ and $|a| \leq |a{+}1|$, hence $C^0(a{+}1) = |a{+}1| \cup l(a{+}1) = |a{+}1| \cup k(a{+}1) = |a{+}1|{+}1 \geq |a|{+}1$, so $\mathrm{eval}_H(a{+}1) > |a|$, that is, $a{+}1 \in \mathrm{NF}_H$. Analogously, $a{+}1 \in \mathrm{NF}_{H'}$. $\tilde{Q} \times \Omega \subseteq \mathrm{NF}_{H'}$ follows from $l_{H'} \restriction (\tilde{Q} \times \Omega) = k \restriction (\tilde{Q} \times \Omega)$. (iv) As $a <' a{+}1$ and $|a| \leq |a{+}1| < \mathrm{eval}_H(a{+}1)$, the claim is by Lemma IV.2.7. (v) If $a \in \tilde{Q} \times \Omega$ is not of the form $a = b{+}n$ for $b = 0$ or $b \in \mathrm{NF}$ a limit, then $a = c{+}n$ for some limit $c \notin \mathrm{NF}_H$, i.e. $c \in L$, the and so $a \oplus 1 \neq a$! (vi) If $a \oplus 1 = a{+}1$, this is by (iii). If $a \oplus 1 = a$, then $a = b{+}n$ for some $b \in \mathrm{NF}_H$, and then $|b| < \mathrm{eval}_H(b)$ implies $|b{+}n| \leq |b|{+}n < \mathrm{eval}_H(b) < \mathrm{eval}_H(a)$. □

Lemma IV.2.17. *Let $\oplus := \oplus_{(\tilde{Q} \times \Omega) \setminus NF_H}$. For all $a \in A$, $eval_{H'}(a) = eval_H(a \oplus 1)$.*

Proof By induction on $<'$, we prove that $C_{H'}(a) = C_H(a \oplus 1)$ by showing by side induction on n: (i) $\forall n C^n_{H'}(a) \subseteq C_H(a \oplus 1)$ and (ii) $\forall n C^n_H(a \oplus 1) \subseteq C_{H'}(a)$.

(i) First, we argue that $C^0_{H'}(a) \subseteq C_H(a \oplus 1)$. If $a \oplus 1 = a{+}1$, then $C^0_{H'}(a) \subseteq |a|{+}1 \leq \mathrm{eval}_H(a{+}1) = C_H(a \oplus 1)$. And if $a \oplus 1 = a$, then either $a \in \Sigma_{asc}$ and $C^0_{H'}(a) = C^0_H(a) \subseteq C_H(a)$, or $a \in \tilde{Q} \times \Omega$ and $a \in \mathrm{NF}_H$, and thus $C^0_{H'}(a) \subseteq |a| \cup k(a) \subseteq C_H(a)$. Note that $C_H(a)$ and $C_H(a \oplus 1)$ are in E. Now,

$$
\begin{aligned}
C^{n+1}_{H'}(a) = \{\mathrm{eval}_{H'}(b) : b <' a \wedge k(b) \subseteq C^n_{H'}(a)\} &\subseteq_{IH} \\
\{\mathrm{eval}_H(b \oplus 1) : b <' a \wedge k(b) \subseteq C_H(a \oplus 1)\} &\subseteq \\
\{\mathrm{eval}_H(b \oplus 1) : b \oplus 1 <' a \oplus 1 \wedge k(b \oplus 1) \subseteq C_H(a \oplus 1)\} &\subseteq \\
\{\mathrm{eval}_H(b) : b <' a \oplus 1 \wedge k(b) \subseteq C_H(a \oplus 1)\} &\subseteq C_H(a \oplus 1).
\end{aligned}
$$

(ii) First, we show that $C^0_H(a) \subseteq C_{H'}(a)$. If $a \oplus 1 = a$, then as $l(a) \subseteq l_{H'}(a)$, $C^0_H(a) \subseteq C^0_{H'}(a)$. And if $a \oplus 1 = a{+}1$, then $a = (x, \alpha) \in \mathrm{NF}_{H'}$, so $C^0_H(a \oplus 1) \subseteq |a|{+}1 < \mathrm{eval}_{H'}(a) = C_{H'}(a)$. For the side-induction step, recall that $C_{H'}(a)$ is an initial segment of the ordinals, and assume that $C^n_H(a \oplus 1) \subseteq C_{H'}(a)$. To see that $C^{n+1}_H(a \oplus 1) = \{\mathrm{eval}_H(b) : b <' a \oplus 1 \wedge k(b) \subseteq C^n_H(a \oplus 1)\} \subseteq C_{H'}(a)$, we need to check that $\mathrm{eval}_H(b) \in C_{H'}(a)$ for each b with $b <' a \oplus 1$ and $k(b) \subseteq C^n_H(a \oplus 1)$. If $b <' a$ (so $a \oplus 1 = a{+}1$), then $\mathrm{eval}_H(b) \leq \mathrm{eval}_H(b \oplus 1) =_{IH} \mathrm{eval}_{H'}(b)$. As $|b \oplus 1| \leq |a \oplus 1| \subseteq C^n_H(a \oplus 1) \subseteq C_{H'}(a)$, also $\mathrm{eval}_{H'}(b) \in C_{H'}(a)$. And if $b = a$, then either $a \notin \mathrm{NF}_H$ and $\mathrm{eval}_H(a) = |a| \in C_{H'}$, or $a = c{+}n{+}1$ for some $c \notin \mathrm{NF}_H$, and by I.H., $\mathrm{eval}_H(a) = \mathrm{eval}_H((c{+}n) \oplus 1) = \mathrm{eval}_{H'}(c{+}n) < \mathrm{eval}_{H'}(a) = C_{H'}(a)$. □

Lemma IV.2.18. *Let $\oplus := \oplus_{(\tilde{Q} \times \Omega) \setminus NF}$. Then $eval_H[Cl_H] = eval_{H'}[Cl_{H'}]$.*

Proof $\mathrm{eval}_H[\mathrm{Cl}_H] \subseteq \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$: else, there is a least $\alpha \in \mathrm{eval}_H[\mathrm{Cl}_H] \setminus \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$. So $\alpha = \mathrm{eval}_H(a)$ for some $a \in \mathrm{Cl}$ and $k(a) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]$. As $a \in \mathrm{NF}_H$, $|a| < \alpha$, hence by I.H. $k(a) \subseteq \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$. Since $a \in \mathrm{NF}_{H'}$ and also $\alpha \leq \mathrm{eval}_H(a \oplus 1) = \mathrm{eval}_{H'}(a)$, $a \in \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$!
$\mathrm{eval}_{H'}[\mathrm{Cl}_{H'}] \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]$: else, there is a least $\alpha \in \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}] \setminus \mathrm{eval}_H[\mathrm{Cl}_H]$. So $\alpha = \mathrm{eval}_{H'}(a)$ for some $a \in \mathrm{Cl}_{H'}$ and $k(a) \subseteq \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$. As $a \in \mathrm{NF}_{H'}$, $|a| < \alpha$, hence by I.H. $k(a) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]$. Since $\alpha = \mathrm{eval}_{H'}(a) = \mathrm{eval}(a \oplus 1)$, and also $a \oplus 1 \in \mathrm{NF}_H$, $a \in \mathrm{eval}_H[\mathrm{Cl}_H]$! $\square$

With the previous two lemmas at hand, we readily obtain the following.

Lemma IV.2.19. *Let $\oplus := \oplus_{(\tilde{Q}\times\Omega)\setminus NF}$. Then, $Cl_H = \{a \oplus 1 : a \in Cl_{H'}\}$.*

Proof First we show that $a \in \mathrm{Cl}_{H'} \Rightarrow a \oplus 1 \in \mathrm{Cl}_H$. Assume that $a \in \mathrm{Cl}_{H'}$. Then $|a| < \mathrm{eval}_{H'}(a) = \mathrm{eval}_H(a \oplus 1)$. Hence, $a \oplus 1 \in \mathrm{NF}_H$, and with $k(a) \subseteq \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$, also $k(a \oplus 1) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]$. Therefore, $a \oplus 1 \in \mathrm{Cl}_H$. For the converse direction, we let $a \in \mathrm{Cl}_H$ and look for a $b \in \mathrm{Cl}_{H'}$ with $b \oplus 1 = a$. If $a = a \oplus 1$, then $|a| < \mathrm{eval}_H(a \oplus 1) = \mathrm{eval}_{H'}(a)$, so $a \in \mathrm{NF}_{H'}$. As with $k(a) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]$ also $k(a \oplus 1) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H] = \mathrm{eval}_{H'}[\mathrm{Cl}_{H'}]$, $a \oplus 1 \in \mathrm{Cl}_{H'}$. And if $a{+}1 = a \oplus 1$, then $a = b{+}n$ for some $b \notin \mathrm{NF}_H$. As $a \in \mathrm{Cl}_H \subseteq \mathrm{NF}_H$, $n = m{+}1$. Hence $a = (b{+}m) \oplus 1$ for $b{+}m \in \mathrm{Cl}_{H'}$. $\square$

Remark IV.2.20. *Let $L' := g'_x \cup \{0\}$ if $o(x)$ is a limit, and else, $L' := g'_x$. Then, for $x \in \tilde{Q}$, $eval_{H'}(x, \alpha) = \tilde{g}_x(\alpha \oplus_{L'} 1)$. Hence $\alpha \mapsto \tilde{g}_x(\alpha \oplus_{L'} 1)$ is monotone and $\tilde{g}_x(\alpha \oplus_{L'} 1) > |(x, \alpha)|$. We regard this function as a fixed point free variant of $\tilde{g}_x$.*

We conclude this section by presenting an OFG $\mathcal{O}_{H^*}$, so that $\mathrm{eval}_{H'} = \mathrm{eval}_{H^*} := \mathrm{eval}_{\mathcal{O}_{H^*}}$. Thereto, we use some notations form Rathjen and Vizcaíno [18]. This allows to relate $\mathrm{eval}_{H^*}\restriction(\tilde{Q} \times \Omega)$ to the ϑ-function in the next section.

Definition IV.2.21. *The (finite) set $E_\Omega(\alpha)$ consists of the ε-numbers below Ω which are needed for the unique representation of α in Cantor normal form.*

(i) $E_\Omega(0) := E_\Omega(\Omega) := \emptyset$,

(ii) $E_\Omega(\gamma) = \{\gamma\}$, *if* $\gamma \in E \cap \Omega$,

(iii) $E_\Omega(\alpha) := E_\Omega(\alpha_k) \cup \ldots \cup E_\Omega(\alpha_1)$, *if* $\alpha =_{NF} \omega^{\alpha_k} + \ldots + \omega^{\alpha_0}$.

Further, $\alpha^* := \mathsf{max}(E_\Omega(\alpha) \cup \{0\})$.

Since ω^α is additively principal, we have the following.

Lemma IV.2.22. *For all $\alpha, \beta < \Omega$, we have that $\alpha \leq \beta \Rightarrow \alpha^* \leq \beta^*$.*

Lemma IV.2.23. *Let $k^*(a) := \{\alpha^* : \alpha \in k(a)\}$, $l^*(a) := \{\alpha^* : \alpha \in l_{H'}(a)\}$, and $\mathcal{O}_{H^*} = (A, <', k^*, l^*)$. Then, $eval_{H'} = eval_{H^*}$, and $NF_{H^*} = NF_{H'}$, $Cl_{H^*} = Cl_{H'}$ and $A[Cl_{H^*}] = A[Cl_{H'}]$.*

Proof First, observe that for $a \in \Sigma_{asc}$, $\mathrm{eval}_{H^*}(a) = G(a)$ (this is shown similar as Lemma IV.2.10). So for $a \in \Sigma_{asc}$, $\mathrm{eval}_{H^*}(a) = \mathrm{eval}_{H'}(a)$. Next, we let $a \in \tilde{Q} \times \Omega$ and show by induction on α: $\alpha^* \in C_{H^*}(a)$ iff $\alpha \in C_{H^*}(a)$. If $\alpha \in E$, then $\alpha = \alpha^*$ and we are done. Else, $a = \langle\rangle$ and the claim holds trivially, or $\alpha =_{NF} \omega^{\beta_k} + \ldots + \omega^{\beta_0}$ with $\beta_0 \leq \ldots \beta_k < \alpha$. Then $b := \langle \beta_0, \ldots, \beta_k \rangle <' a$. If $\alpha^* \in C_{H^*}(a)$, then as $\alpha^* = \beta_k^*$, also $\beta_0^*, \ldots, \beta_k^* \in C_{H^*}(a)$, so by I.H., $\vec{\beta} \in C_{H^*}(a)$, that is $|b| \in C_{H^*}(a)$, hence $\mathrm{eval}_{H^*}(b) = \alpha \in C_{H^*}(a)$.
Replacing H^* by H' in the above argument yields that for each $a \in A$, $\alpha^* \in C_{H'}(a)$ iff $\alpha \in C_{H'}(a)$. Now the $C_{H'}(a) = C_{H^*}(a)$ is immediate by the definition of these sets. $\square$

IV.3 How to obtain a notation system for $G[\mathrm{Cl}_H]$

In this section, we sketch how to define notations for the ordinals in $G[\mathrm{Cl}_H]$. Essentially by course of value recursion, we define a primitive recursive well-ordering $(O, \lhd)$ that is order-isomorphic to $(\mathrm{eval}_H[\mathrm{Cl}_H], \in)$. It is however convenient to actually have an ordering $(O', \unlhd)$ that is order-isomorphic to $(A[\mathrm{Cl}_H], \preceq)$, where $a \preceq b :\Leftrightarrow \mathrm{eval}_H(a) \leq \mathrm{eval}_H(b)$.

First, we consider a slightly more general form of course of value recursion. Towards its formulation, let $[x, y] := \frac{1}{2}(x+y)(x+y+1)+y$ be Cantor's pairing function with associated projections $[\cdot]_0$ and $[\cdot]_1$, that is, if $z = [x, y]$, then $[z]_0 = x$ and $[z]_1 = y$. Recall that $[\cdot, \cdot]$ is bijective and monotone in both arguments. Further, $[0, 0] = 0$.

The course of value of a function $f(n)$ is the function $\overline{f}(n) := \langle f(0), \ldots, f(n-1) \rangle$. Then, with $g(n)$ also the function $f(n)$ with $f(n) = g(\overline{f}(n), x)$ is primitive recursive. When working with binary functions (characteristic functions of binary relations), it is convenient to consider the following form of course of value recursion.

Definition IV.3.1. *Let $\overline{f}(0,0) := \langle\rangle$ and for $\{x, y\} \neq \{0\}$,*

$$\overline{f}(x, y) := \langle f([0]_0, [0]_1), \ldots, f([n]_0, [n]_1) \rangle, \quad \textit{where } n := [x, y]-1.$$

Lemma IV.3.2. *Assume that $g(x)$ is primitive recursive. Then also the function $f(x, y)$ with $f(x, y) = g(\overline{f}(x, y), x, y)$.*

Similar to standard course of value recursion, we can also define functions by simultaneous recursion. We can of course combine this with standard course of value recursion.

Lemma IV.3.3. *If $g_i(\vec{x})$ $(0 \leq i \leq k{+}l)$ are $k{+}l{+}2$ary primitive recursive functions, then also f_i and h_j $(0 \leq i \leq k,\ 0 \leq j \leq l)$ with*

(i) $f_i(x,y) = g_i(\overline{f}_1(x,y),\ldots,\overline{f}_k(x,y),\overline{h}_1([x,y]),\ldots,\overline{h}_l([x,y]),x,y)$,

(ii) $h_j([x,y]) = g_j(\overline{f}_1(x,y),\ldots,\overline{f}_k(x,y),\overline{h}_1([x,y]),\ldots,\overline{h}_l([x,y]),x,y)$.

Using such a simultaneous recursion, we define codes for ordinals and names, and orderings on names and ordinals. In order to keep codes of names distinct from codes of ordinals, we use two different kinds of sequence numbers. Namely, $\langle\ldots\rangle$, which yields even numbers, and $(\ldots)$, which yields odd numbers. Below, $p : \mathbb{N} \to \mathbb{N}$ is the primitive recursive function that enumerates the prime numbers.

Definition IV.3.4. *For each $n \in \mathbb{N}$, $\langle\ldots\rangle_n : \mathbb{N}^n \to \mathbb{N}$ and $(\ldots)_n : \mathbb{N}^n \to \mathbb{N}$, where $\langle\rangle := 0$, $() := 1$, and*

(i) $\langle x_0,\ldots,x_{n-1}\rangle_n := 2\cdot\Pi_{i<n}p(i)^{x_i+1}$ *and* $(x_0,\ldots,x_{n-1})_n := 2\cdot\Pi_{i\leq n+1}p(i)^{x_i+1}{+}1$,

(ii) $\mathsf{lh}(\langle x_0,\ldots,x_{n-1}\rangle_n) := \mathsf{lh}((x_0,\ldots,x_{n-1})_n) := n$,

(iii) $\pi((\langle x_0,\ldots,x_{n-1}\rangle_n),i) := \pi((x_0,\ldots,x_{n-1}\rangle_n),i) := x_i$ *$(i<n)$.*

$\mathsf{seq}_{\langle\rangle} := \bigcup_n \mathsf{rng}(\langle\ldots\rangle_n)$ *and* $\mathsf{seq}_{()} := \bigcup_n \mathsf{rng}((\ldots)_n))$, *and we write $(x)_i$ for $\pi(x,i)$. For both kinds of sequences, concatenation is denoted by $*$, e.g. $(a,b)*(c) = (a,b,c)$.*

We use $(\cdots)$ to code names, so names are coded by odd numbers, and $\langle\cdots\rangle$ to code ordinals, so ordinals are coded by even numbers. $\langle\rangle = 0$ is a code of the ordinal 0, and $() := 1$ is a code for the name q_0. Further, we use 3 as a code for -1.

Below, we also make use of finite sets. The idea is that they are coded as finite sequences. If X is a finite set (or a finite sequence respectively), then $X < a$ states that for each $x \in X$, $x < a$. Further, we let σ,τ range over elements of $\mathsf{seq}_{\langle\rangle}$, and $x,y,\ldots$ over elements of $\mathsf{seq}_{()}$.

Next, we define primitive recursive subsets O'', Q'', Σ''_{asc} and L'' of the natural numbers. The idea is that O'' is a set that has the structure of the codes O' of $A[\mathrm{Cl}]$, $\Sigma''_{asc} \subseteq O''$ has the structure of (the codes of) Σ_{asc}, Q'' has he structure of Q^H, and L'' the structure of limit ordinals. Then, we define primitive recursive functions $\hat{o}$, $\widehat{\deg}$, $\hat{k}$, $\widehat{\mathrm{length}}$, which do on codes what the functions o, deg, k, length do on names and ordinals, respectively. It is readily seen that these sets and the functions are primitive recursive.

Definition IV.3.5. *By simultaneous recursion, we define the set O'', Q'', Σ''_{asc}, L'', and the functions $\hat{o}$, $\widehat{\deg}$, $\hat{k}$ and $\widehat{length}$.*

(i) $q_0 \in Q''$, $0 \in \Sigma''_{asc}$ *and* $0 \in O''$. *Further,* $\hat{o}(q_0) := 1$, *and* $\widehat{\deg}(q_0) := 0$.

(ii) *If* $a_0, \ldots, a_k \in O''$*, then* $\sigma := \langle a_0, \ldots, a_k \rangle \in \Sigma''_{asc}$ *and* $\sigma \in O''$*, and if* $a_0 \neq 0$*, then* $\langle a_0, \ldots, a_k \rangle \in L''$*.*

(iii) *If* $a_0, \ldots, a_k \in O''$ *and* $x_0, \ldots, x_k \in Q''$*, then* $((a_0, x_0), \ldots, (a_k, x_k)) \in Q''$*.*

(iv) *If* $x = ((a_0, x_0), \ldots, (a_k, x_k))$ *or* $x \notin Q''$*, then* $\hat{o}(x) := \widehat{\deg}(x) := 0$*. Else, if* $x = ((a_0, x_0), \ldots, (a_k, x_k))$ *and* $a_0 \in L''$*, then* $\hat{o}(x) := a_0$ *and* $\widehat{\deg}(x) := 1$*; if* $a_0 \notin L''$*, then* $\hat{o}(x) := \hat{o}(x_0)$ *and* $\widehat{\deg}(x) = \widehat{\deg}(x_0)+1$*.*

(v) *If* $x \in Q''$ *and* $a \in O''$*, then* $\langle x, a \rangle \in O''$ *and* $\langle x, a \rangle \in L''$*.*

(vi) $\hat{k}(q_0) = \emptyset$*,* $\hat{k}(\langle a_1, \ldots, a_k \rangle) := \hat{k}(a_1) \cup \ldots \cup \hat{k}(a_k)$*,* $\hat{k}(x, a) = \hat{k}(x) \cup \{a\}$*, and* $\hat{k}((x_1, \ldots, x_k)) := \hat{k}(x_1) \cup \ldots \cup \hat{k}(x_k)$*. Otherwise,* $\hat{k}(m) = \emptyset$*.*

(vii) $\widehat{length}(0) := \widehat{length}(q_0) := 0$*,*
for $0 \neq a \in O''$*,* $\widehat{length}(a) := \mathsf{max}(\{\widehat{length}(a) : a \in \hat{k}(a)\})+1$*, and*
for $1 \neq x \in Q''$*,* $\widehat{length}(x) := \mathsf{max}(\{\widehat{length}(a) : a \in \hat{k}(x)\})+1$*.*

Next, we define an auxiliary function $x \mapsto x[0]$, and then $\hat{l} : \mathsf{N} \to \mathsf{N}$.

Definition IV.3.6. *First, let* $x \mapsto x[0]$ *so that for each* $x = ((\alpha_0, x_0), \ldots, (\alpha_k, x_k))$*,*

(i) *if* $\hat{o}(x) \in L'' \wedge \alpha_0 \in L''$*, then* $x[0] := ((1, x_0), \ldots, (\alpha_k, x_k))$*,*

(ii) *if* $\hat{o}(x) \in L'' \wedge \alpha_0 = 1$*, then* $x[0] := ((\alpha_0, x_0[0]), \ldots, (\alpha_k, x_k))$

(iii) *if* $\hat{o}(x) \in L'' \wedge \alpha_0 = \beta+1 \wedge \beta \neq 0$*, then* $x[0] := ((1, x_0[0]), (\beta, x_0), \ldots, (\alpha_k, x_k))$*,*

(iv) *and else,* $x[0] := x$*.*

Secondly, let $\hat{l} : \mathsf{N} \to \mathsf{N}$ *so that*

(i) *if* $\mathsf{lh}(\sigma) = 1 \wedge (\sigma)_0 \in L''$*, then* $(\sigma) := \emptyset$*, and else,* $\hat{l}(\sigma) := \hat{k}(\sigma)$*.*

(ii) $\hat{l}(x) := \hat{k}(x)$*,*

(iii) *if* $a \in L''$*, then* $\hat{l}(\langle x, a \rangle) = \hat{k}(x)$*,*

(iv) *if* $\hat{o}(x) \in L'' \wedge a = 0$*, then* $\hat{l}(\langle x, a \rangle) = \hat{k}(x[0]) \cap \hat{k}(x)$*,*

(v) *if* $\hat{o}(x) \notin L'' \vee 0 \neq a$*, then* $\hat{l}(\langle x, a \rangle) = \hat{k}(\langle x, a \rangle)$*.*

Below, we define the primitive recursive ordering $(O', \lhd')$. It is defined by a simultaneous recursion, in which course we also define other primitive recursive sets and orderings. The names for these sets and relations are overloaded, except for O', O, $\lhd$ and $\simeq$. O' (in contrast to O) contains also codes of elements in $A[\mathrm{Cl}_H] \setminus \mathrm{NF}_H$, and $a \simeq b$ expresses that a and b are codes of elements in $A[\mathrm{Cl}_H]$ that evaluate to the same ordinal.

Further, we use the following abbreviations. Below, $\tilde{Q}$, Q^H and Σ_{asc} are overloaded and denote the sets of the codes of the corresponding names and ordinals.

(i) $x \in \tilde{Q}$ is short for $x \in Q^H \wedge (x \neq q_0 \vee (\widehat{\deg}(x) > 1 \rightarrow \hat{o}(x) \in L'')$.

(ii) $a \in O$ is short for $a \in O' \wedge k(a) \lhd a$.

(iii) $a \unlhd b$ is short for $a \lhd b \vee a \simeq b$, where $a \simeq b$ is defined below so that $a \simeq b$ iff a and b code the same ordinal. Hence if $a, b \in O$, then $a \simeq b$ iff $a = b$.

(iv) $a \in \Sigma_{asc}$ is short for $a \in \mathsf{seq}_{\langle\rangle}$, and $a \in Q' \times O$ is short for $(a)_0 \in Q' \wedge (a)_0 \in O$.

Further, recall that for any ordering $<$, the derived ordering $<_{\text{lex}}$ orders finite sequences as follows (cf. Definition II.1.1): $\sigma := \langle a_0, \ldots a_k \rangle <_{\text{lex}} \langle b_0, \ldots b_l \rangle =: \tau$ iff there is an $i \leq \mathsf{min}\{k+1, l\}$ so that $\langle a_{k+1-i}, \ldots a_k \rangle = \langle b_{l+1-i}, \ldots b_l \rangle$ and further, $i \leq k \rightarrow a_{k-i} < b_{l-i}$.

Definition IV.3.7. *By simultaneous recursion, we define (among other sets and relations) the set O', Q^H, $\lhd$, $\simeq$, $<^H$ and $<'$, where Q^H, $<^H$ and $<'$ get overloaded.*

(i) $\langle\rangle \in \Sigma_{asc}$, $\langle\rangle \in O'$ *and* $() \in Q^H$.

(ii) If $a_0, \ldots, a_k \in O$ *and* $a_0 \unlhd \ldots \unlhd a_k$, *then* $\sigma := \langle a_0, \ldots, a_k \rangle \in \Sigma_{asc} \subseteq O'$.

(iii) If $a_0, \ldots, a_k \in O$ *and* $x_0, \ldots, x_k \in Q^H$ *and* $x_1 <^H \ldots <^H x_k$, *then* $x := ((\alpha_0, x_0), \ldots, (\alpha_k, x_k)) \in Q^H$.

(iv) If $x \in \tilde{Q}$ *and* $a \in O$, *then* $\langle x, a \rangle \in O'$.

(v) $\widehat{|a|} := 3$ *(the code for* -1*) if* $\hat{k}(a) = \emptyset$ *and else,* $\widehat{|a|} := \mathsf{max}_{\lhd}(\hat{k}(a))$.

(vi) $(a, x) \lessdot (b, y)$ *iff* $x <^H y \vee (x = y \wedge a \lhd b)$.

(vii) $x <^H y$ *iff* $x \lessdot_{lex} y$.

(viii) $\sigma <' \tau$ *iff* $\sigma \lhd_{lex} \tau$, $\langle x, a \rangle <' \langle y, b \rangle$ *iff* $x <^H y \vee x = y \wedge a \lhd b$, *and* $\sigma <' (x, a)$.

(ix) $a \lhd b :\Leftrightarrow (a <' b \wedge \widehat{|a|} \lhd b) \vee a \lhd \widehat{|b|} \vee a \in \hat{l}(b))$, *and* $3 \lhd z$ *iff* $a \neq 3$.

(x) $a \simeq b$ *iff* $a <' b \wedge b \simeq |\widehat{a}| \wedge b \notin \hat{l}(a)$ *or* $b <' a \wedge a \simeq |\widehat{b}|$ *or* $a = b$.

Next, we assign to each $a \in O'$ and ordinal expression, and to each $x \in O'$ a name.

Definition IV.3.8. *For each $a \in O'$ and each $x \in O'$, define $f(a)$ and $f(x)$ as follows.*

(i) $f(\langle\rangle) := 0$ *and* $f(()) := q_0$,

(ii) $f((\alpha_1, x_1), \ldots, (\alpha_k, x_k)) := \langle (f(\alpha_1), f(x_1)), \ldots (f(\alpha_1), f(x_1)) \rangle$.

(iii) $f(\langle a_1, \ldots, a_k \langle) := \langle f(\alpha_1), \ldots, f(\alpha_k) \rangle$,

(iv) $f(\langle x, a \rangle) := (f(x), f(a))$.

The following is then readily checked.

Lemma IV.3.9. $f : (O', \trianglelefteq) \to (A[Cl_H], \preceq)$, *where* $a \preceq b :\Leftrightarrow eval_H(a) \leq eval_H(b)$, *and* $eval_H \circ f : (O, \lhd) \to (eval_H[Cl_H], \in)$ *are order-isomorphisms.*

Proof First, we let $\tilde{Q}[\mathrm{Cl}_H] := \{x \in \tilde{Q} : k(x) \subseteq \mathrm{eval}_H[\mathrm{Cl}_H]\}$, and extend the function $\mathrm{length} : A[\mathrm{Cl}_H] \to \mathbb{N}$ (cf. Definition IV.2.8) to names as well by setting $\mathrm{length}(q_0) := 0$, and if $0 \neq x$ and $x \in \tilde{Q}[\mathrm{Cl}_H]$, then $\mathrm{length}(x) := \mathsf{max}(\mathrm{length}[k^0(x)]) + 1$.

Now, an easy induction on $\widehat{\mathrm{length}}(a)$ yields that for each $a \in (O' \cup \tilde{Q})$, $\widehat{\mathrm{length}}(a) = \mathrm{length}(f(a))$, and further, $\hat{k}(v) = k(f(v))$ and $\hat{l}(v) = l(f(v))$. Another easy induction on $\widehat{\mathrm{length}}(a)$ then yields that for each $a \in (O' \cup \tilde{Q})$, $\widehat{\mathrm{length}}(a) = \widehat{\mathrm{length}}(b) \wedge f(a) = f(b)$ implies $a = b$, thus $f : O' \cup \tilde{Q} \to A[\mathrm{Cl}_H] \cup \tilde{Q}[\mathrm{Cl}_H]$ is injective. That f is also surjective is shown by induction on the length $\mathrm{length}(w)$ of the output $w \in A[\mathrm{Cl}] \cup \tilde{Q}[\mathrm{Cl}_H]$.

By a simultaneous induction on $\widehat{\mathrm{length}}(a) + \widehat{\mathrm{length}}(b)$ and $\widehat{\mathrm{length}}(x) + \widehat{\mathrm{length}}(y)$ one next shows that $a \lhd b$ iff $f(a) < f(b)$, $x <^H y$ iff $f(x) <^H f(y)$, and $a <' b$ iff $f(a) <' f(b)$. Then, we also have that $a \in O$ (i.e. $a \in O' \wedge |\widehat{a}| \lhd a$), iff $f(a) \in A[\mathrm{Cl}_H] \wedge k(f(a)) < f(a)$ iff $f(a) \in \mathrm{Cl}_H$.

This yields that $f : (O', \trianglelefteq) \to (A[\mathrm{Cl}_H], \preceq)$ is an order-isomorphism. As by Lemma IV.2.7, $\mathrm{eval}_H : (\mathrm{Cl}_H, \prec) \to (\mathrm{eval}_H[\mathrm{Cl}_H], \in)$ is an order-isomorphism, the second claim follows as well. $\square$

It is not hard, but a bit cumbersome to turn this into a full-fledged notation system.

IV.4 Finding notations for the ordinals $g_x(\alpha)$

The notation system is based on the functions $\tilde{g}_x$, however, in our proofs in Chapter II and III we used the functions g_x instead. Therefore, we need to know what notation corresponds to the ordinal $g_x(\alpha)$: given $(x,\alpha) \in Q^H_{N_0} \times \Omega$, we need an ordinal expression $v(x,\alpha) \in A[\mathrm{Cl}]$ so that $g_x(\alpha) = G_v(x,\alpha) := G(v(x,\alpha))G(v(x,\alpha))$, where $G : A \to \Omega$ is the function form DefinitionIV.2.2, and Cl and eval are w.r.t. the OFG $\mathcal{O}_H = (A, <', k, l)$ from subsection IV.2.1.
Further, the (coded version of the) function v should already be provably total in ACA_0. Moreover, ACA_0 should be able to prove the following for each $\alpha, \beta \in \mathrm{eval}[\mathrm{Cl}]$, and $x, y \in Q^H_{N_0}$ with $(x,0),(y,0) \in A[\mathrm{Cl}]$ and $o(y) = \gamma_0$:

(i) $\alpha < \beta$ implies $G_v(x,\alpha) < G_v(x,\beta)$,

(ii) $G_v(x,\gamma) = \mathsf{sup}_{\alpha<\gamma} G_v(x,\alpha)$, and

(iii) for each $\gamma \leq \gamma_0$, $G_v(y[\delta_0+\gamma], \alpha+1) = \mathsf{sup}_{\xi \lhd \gamma} s_\xi$, where $s_0 = G_v(y[\delta_0+\gamma_0], \alpha)+1$, $s_{\xi+1} := G_v(y[\delta_0+\xi], s_\xi)$ and for $\gamma' < \gamma_0$, $s_{\gamma'} := G_v(x[\delta_0+\gamma'], s_0) = \mathsf{sup}_{\xi<\gamma'} s_\xi$.

These properties are used in the proofs of Lemma III.7.9 and III.7.10 (ii), and the corresponding results in Chapter II. The reason why these properties are required is that ACA_0 does not know that g_x is normal. Within ACA_0 we cannot argue using properties of the functions $(g_x : x \in Q^H)$. We just know how to compare notations $v(x,\alpha)$ and $v(y,\beta)$ which correspond to the ordinals $\tilde{g}_x(\alpha)$ and $\tilde{g}_y(\beta)$ (where $\tilde{g}_x = g_x$ if $o(x) \in \mathsf{Lim}(\Omega)$ or $\deg(x) > 1$ or $x = q_0$, and $\tilde{g}_{y+1}(\alpha) = g_x(\omega(1+\alpha))$.

It is convenient to extend G to $G : A \cup (\{q_0\} \times \Omega)$ by setting $G(q_0,\alpha) := g(\alpha)$. Further, for this subsection, we let $\Omega := \mathrm{eval}[\mathrm{Cl}] = G[\mathrm{Cl}]$ and $\alpha, \beta, \ldots, \gamma$ range over eval[Cl] (now Ω), where again, γ, γ', possibly with subscripts, range over limit ordinals. Moreover, for $a, b \in A$, we let $a \prec b :\Leftrightarrow G(a) < G(b)$. Note the by regarding Ω as eval[Cl], A becomes $A[\mathrm{Cl}]$ and Q' becomes $\{x \in Q' : k(x) \subseteq \mathrm{eval}[\mathrm{Cl}]\} = \{x \in Q' : (x,0) \in A[\mathrm{Cl}]\}$.

In order to ensure that the function $v : Q^H_{N_0} \times \Omega \to \Omega$ is provably total in ACA_0, we define it by recursion on $\|(x,\alpha)\|_t$, where (for $N_0 > 1$) $\|\cdot\|_t : Q^H_{N_0} \times \Omega \to \omega_{N_0}$ is defined below.

Definition IV.4.1. *Let $N_0 > 0$. First, we define $|\cdot|_Q : Q^H_{N_0} \to \omega_{N_0}$ as follows: if $x = q_0 \vee o(x) \in \mathsf{Lim}(\Omega)$, then $|x|_Q := 0$, and otherwise,*

(i) for $n > 0$, $|(n,x)|_Q := \omega^{|x|_Q} n$ and $|(\gamma+n,x)|_Q := \omega^{|x|_Q} n$,

(ii) if $k > 1$, then $|\langle x_1, x_2, \ldots, x_k\rangle|_Q = |x_1|_Q + |\langle x_2, \ldots, x_k\rangle|_Q$.

Now, we define $\|\cdot\|_t : Q^H_{N_0} \times \Omega \to \omega_{N_0}$ *as follows: if* (x, α) *is good (cf. Definition IV.1.6), then* $\|(x, n)\|_t := 0$*, and else* $\|(x, n)\|_t := |x|_Q{+}n$ *and* $\|(x, \gamma{+}n)\|_t := |x|_Q{+}n$.

The following properties of $\|\cdot\|$ and $\|\cdot\|_t$ are readily checked.

Lemma IV.4.2. *For each* $(x, \alpha) \in Q^H_{N_0} \times \Omega$,

(i) If $|x|_Q = 0$*, then* $x = q_0$ *or* $o(x) \in \mathsf{Lim}(\Omega)$.

(ii) $\|(q_0, \alpha)\|_t < \omega$*, and if* $N_0 > 0$*, then* $\|(x, \alpha)\|_t < \omega_{N_0}$*,*

(iii) if $\deg(x) > 1$ *and* $o(x) = 1$*, then* $\|(x[1{+}\alpha], 0)\|_t < \|(x, \alpha)\|_t$*,*

(iv) for all γ, γ' *and* α*,* $\|(x, \gamma)\|_t < \|(x{+}1, 0)\|_t$*,* $\|(x, \gamma'{+}1)\|_t < \|(x{+}1, \gamma{+}1)\|_t$*, and* $\|(x{+}1, \alpha{+}1)\|_t < \|(x{+}1, \alpha{+}2)\|_t$.

Next, we define $v : Q^H \times \Omega \to A[\mathrm{Cl}]$ by recursion on $\|\cdot\|_t$ so that $g_x(\alpha) = G_v(x, \alpha)$.

Definition IV.4.3. $v : Q^H \times \Omega \to A[Cl]$ *is defined as follows.*

(i) $v(q_0, \alpha) = \langle 1{+}\alpha \rangle$*,* $v(x{+}1, \omega(1{+}\alpha)) = (x{+}1, \alpha)$*, and if* $o(y) \in \mathsf{Lim}(\Omega)$*, then* $v(y, \alpha) := (y, \alpha)$.

(ii) If $\deg(z) > 1$ *and* $o(z) = 1$*, then* $v(z, \alpha) = v(z[1{+}\alpha], 0)$.

(iii) $v(x{+}1, 0) = v(x, G(v(x, 0)))$*,* $v(x{+}1, \gamma{+}1) = v(x, G(v(x{+}1, \gamma)){+}1)$ *and* $v(x{+}1, \alpha{+}2) = v(x, G(v(x{+}1, \alpha{+}1)))$.

It is in general not obvious how $v(x, \alpha)$ and $v(y, \beta)$ compare. However, if the ordinal argument is a limit, the following cases are quite easy.

Lemma IV.4.4. *Let* $(x, \alpha) \in Q^H \times \Omega$*. Then we have the following.*

(i) $v(x, \omega(1{+}\alpha)) \prec v(x, \omega(1{+}\beta))$ *iff* $\alpha < \beta$.

(ii) $v(x, \omega(1{+}\alpha)) \preceq v(x{+}1, \omega(1{+}\alpha))$*, and* $v(x, \omega(1{+}\alpha)) \prec v(x{+}1, \omega(1{+}\alpha))$ *iff* $\alpha < G(x{+}1, \alpha)$.

(iii) If $\gamma := G(x{+}1, \alpha)$*, then* $G(v(x, \gamma)) = \gamma$.

Proof All claims follows by applying the definition of v and then Lemma IV.2.7 (v). $\square$

To learn more about v, we start with a couple of simple observations.

Lemma IV.4.5. *Let $(x, \gamma) \in Q^H \times \Omega$, $a \in A[Cl]$, and $z \in Q^H$ with $\deg(z) > 1$ and $o(z) = 1$.*

(i) If $v(x, \alpha) \neq (x, \alpha)$, then $v(x, \alpha) <' (x, \alpha)$.

(ii) $v(x, \gamma) \preceq (x, \gamma) \preceq v(x, \omega\gamma)$. Hence if $\omega^\omega \leq \gamma \in HZ$, then $\omega\gamma = \gamma$ and thus $G_v(x, \gamma) = G(x, \gamma)$.

(iii) If $\alpha < \beta$ and $\gamma' < \gamma$, then $v(x{+}1, \gamma') \prec v(x{+}1, \gamma)$, $v(z, \gamma') \prec v(z, \gamma)$, and $v(z[\gamma'], \alpha) \prec v(z[\gamma], \beta)$.

Proof (i) By induction on $\|(x, \alpha)\|_t$. (ii) and (iii) are immediate by the definition of v and Lemma IV.2.7. □

Lemma IV.4.6. *Let $(x, \gamma) \in Q^H \times \Omega$, $a \in A[Cl]$.*

(i) If $x \rightsquigarrow^* y$, then $x < y$ and $|x| \leq |y|$.

(ii) If $x \rightsquigarrow^* y$, then $v(x, \omega) \prec v(y, \omega)$.

(iii) If $o(z) = \gamma$, then $y \rightsquigarrow^* z$, then $v(y, \omega) \prec (z, 0)$.

Proof (i) It suffices to check that $x \rightsquigarrow y$ entails $x < y$ and $|x| < |y|$, which is straightforward. (ii) By (i) and Lemma IV.2.7. (iii) Assume that $o(z) = \gamma$ and $y \rightsquigarrow^* z$. Then by (i), $(y, \omega) <' (z, 0)$ and $|(y, \omega)| \leq |(z, 0)| \leq G(z, 0)$. In case that $|(z, 0)| = G(z, 0)$, then $G(z, 0) = \gamma$ and $\gamma \notin l(z, 0) = k(z[0])$. So for each $\alpha < \gamma$, $|z[\alpha]| < \gamma$. As $y \rightsquigarrow^* z[\alpha]$ for some $\alpha < \gamma$, also $|(y, \omega)| \leq |(z[\alpha], \omega)| < \gamma$. Thus in any case, $|(y, \omega)| < G(z, 0)$, so $v(y, \omega) \preceq (y, \omega) \prec (z, 0)$. □

With the above auxiliary result at hand, the following is readily shown by induction on $|x|_Q$.

Lemma IV.4.7. *Let $\delta \in \mathsf{Lim}(\Omega) \cup \{0\}$. Then $v(x, \delta{+}n) \prec v(x, \delta{+}\omega)$.*

Proof By induction on $|x|_Q$. This is immediate by Lemma IV.2.7 if $x = q_0$ or $o(x) = \gamma$. If $\deg(x) > 1$ and $o(x) = 1$, then $v(x, \delta{+}n) = v(x[\delta{+}n], 0) \prec_{IH} v(x[\delta{+}n], \omega) \prec v(x[\delta{+}\omega], 0)$. And if $x = y{+}1$ we show the claim by side-induction on n. For the case $n = 0$ and $\delta = 0$, let $\gamma := G_v(y, 0)$ and $\gamma' := G_v(y, \omega)$. By the main I.H., $\gamma < \gamma'$, and further, by Lemma IV.4.4 (ii), $\gamma'' := G_v(y{+}1, \omega) > \gamma'$. Hence, $v(y{+}1, 0) = v(y, \gamma) \prec v(y, \gamma') \prec v(y, \gamma'')$, and by Lemma IV.4.4 (iii), $G_v(y, \gamma'') = \gamma''$. If $n = 0$ and $\delta \in \mathsf{Lim}$, then the claim is obvious, and if $n = 1$ and $\delta \in \mathsf{Lim}$, then for $\gamma' := G_v(y{+}1, \delta)$, $\gamma' < G_v(y{+}1, \delta{+}\omega) \in E$, thus also $\gamma'{+}\omega < G_v(y{+}1, \delta{+}\omega)$. Therefore $|v(y, \gamma'{+}\omega)| < G_v(y{+}1, \delta{+}\omega)$, and thus $v(y, \gamma'{+}\omega) \prec v(y{+}1, \delta{+}\omega)$. Now $G_v(y{+}1, \delta{+}1) = G_v(y, \gamma'{+}1) <_{IH} G_v(y, \gamma'{+}\omega) < G_v(y{+}1, \delta{+}\omega)$ follows.

For the side induction step, note that $v(y{+}1, \delta{+}n{+}1) = v(y, v(y{+}1, \delta{+}n))$. By side I.H., $v(y{+}1, \delta{+}n) \prec v(y{+}1, \delta{+}\omega)$, hence by the main I.H., $v(y, G_v(y{+}1, \delta{+}n)) \prec v(y, G_v(y{+}1, \delta{+}\omega))$. The claim follows as $G_v(y, G_v(y{+}1, \delta{+}\omega)) = G_v(y{+}1, \delta{+}\omega)$ by Lemma IV.4.4 (iii). □

For the next lemma, we need a slight generalization of the auxiliary Lemma IV.4.6. Observe that $v(q_0, \omega) = v(q_1, 0) = \langle\omega\rangle$. Further, we have the following.

Lemma IV.4.8. *If $x, y \in Q^H_{N_0}$ and $q_0 \rightsquigarrow^* x \rightsquigarrow^* y$, then $v(x, \omega) \prec v(y, 0)$.*

Proof By induction on $|y|_Q$. Let $q_0 \rightsquigarrow^* x \rightsquigarrow^* y$. If $|y|_Q = 0$, then $o(y) = \gamma$, so $v(y, 0) = (y, 0)$, and $v(x, \omega) \prec v(y, 0)$ is by Lemma IV.4.6 (iii). If $y = z{+}1$, then $v(y, 0) = v(z, \gamma)$ for $\gamma := G_v(z, 0)$. If $x = z$, then $\omega < \gamma$, and so $v(x, \omega) = v(z, \omega) \prec v(z, \gamma) = v(y, 0)$. And if $x \neq z$, then $x \rightsquigarrow^* z$, and by Lemma IV.4.6 (ii), $v(x, \omega) \preceq v(z, \omega) \prec v(z, \gamma) = v(y, 0)$. Finally, if $o(y) = 1$ and $\deg(y) > 1$, then $v(y, 0) = v(y[1], 0)$. As $|y[1]|_Q < |y|_Q$ and $y[0] \rightsquigarrow^* y[1]$, the I.H. yields $v(y[0], \omega) \prec v(y[1], 0)$. As $x \rightsquigarrow^*_r y[0]$, $v(x, \omega) \preceq v(y[0], \omega)$ by Lemma IV.4.6 (ii). Therefore, $v(x, \omega) \prec v(y[1], 0) = v(y, 0)$. □

Lemma IV.4.9. *Let $\delta \in \mathsf{Lim}(\Omega) \cup \{0\}$ and $n > 0$. Then $v(x, \delta) \prec v(x, \delta{+}n)$.*

Proof By induction on $|x|_Q$. This is clear if $x = q_0$. If $o(x) = \gamma$, then $v(x, \delta) = (x, \delta)$ and $v(x, \delta{+}n{+}1) = (x, \delta{+}n{+}1)$, and the claim is by Lemma IV.2.7. If $\deg(x) > 1$ and $o(x) = 1$, then by Lemma IV.4.8 and IV.4.7, $v(x, \delta) = v(x[1{+}\delta], 0) \prec_{IH} v(x[1{+}\delta], \omega) \prec v(x[1{+}\delta{+}n], 0) = v(x, \delta{+}n)$. Finally, if $x = y{+}1$, then we show the claim by side-induction on $n \geq 1$. First, we consider the case $\delta = 0$. $v(y{+}1, 1) = v(y, \gamma')$ for $\gamma' := G_v(y, 0)$ and by Lemma IV.4.7, $v(y, 0) \prec v(y, \omega) \preceq v(y, \gamma')$. And in case that $\delta \in \mathsf{Lim}(\Omega)$ is a limit, then for $\gamma' := G_v(y{+}1, \delta)$, $v(y{+}1, \delta{+}1) = v(y, \gamma'{+}1) \succ_{IH} v(y, \gamma') = \gamma'$ by Lemma IV.4.4 (iii). For the side-induction step, observe that for $\gamma' := G_v(y{+}1, \delta{+}n)$, $v(y{+}1, \delta) \prec_{IH} v(y{+}1, \delta{+}n) \preceq v(y, \gamma') = v(y{+}1, \delta{+}n{+}1)$. □

An easy induction on n now yields the following.

Lemma IV.4.10. *Let $\delta \in \mathsf{Lim}(\Omega) \cup \{0\}$. Then $v(x{+}1, \delta{+}n) \prec v(x{+}1, \delta{+}n{+}1)$.*

Proof By induction on n. We just consider the case $\delta = \gamma$. If $n = 0$, $v(x{+}1, \delta) \prec v(x{+}1, \delta{+}1)$ is by Lemma IV.4.7. The induction step: $v(x{+}1, \gamma{+}n{+}2) = v(x, \gamma')$ for $\gamma' := G_v(x{+}1, \gamma{+}n{+}1))$. By I.H., $\gamma'' := G_v(x{+}1, \gamma{+}n)) < \gamma'$, so also $\gamma''{+}1 < \gamma'$. Hence, using Lemma IV.4.4 (ii), $v(x{+}1, \gamma{+}n{+}1) \preceq v(x, \gamma''{+}1) \prec v(x{+}1, \gamma') = v(x{+}1, \gamma{+}n{+}2)$. □

Finally, we show that notations $v(x, a)$ also behave as expected when it comes to taking supremas.

Lemma IV.4.11. *Let $x, y \in Q^H$, where $o(y) = \omega(1+\alpha)$. Then,*

(i) $G_v(x+1, \omega(1+\alpha)) = G(x+1, \alpha) = \mathsf{sup}_{\xi<\omega(1+\alpha)} G_v(x+1, \xi)$, *and*

(ii) $G_v(y, 0) = G(y, 0) = \mathsf{sup}_{\xi<\omega(1+\alpha)} G_v(y[\xi], 0)$.

Proof (i) If $\alpha \in \mathsf{Lim}(\Omega)$, then in view of Lemma IV.4.7 and IV.4.9, we have that $\mathsf{sup}_{\xi<\omega(1+\alpha)} G_v(x+1, \xi) = \mathsf{sup}_{\xi<\alpha} G_v(x+1, \omega(1+\xi)) = \mathsf{sup}_{\xi<\alpha)} G(x+1, \xi) = (x+1, \alpha)$. So it remains to deal with the case where $\alpha = 0$ or $\alpha = \beta+1$. We show by induction on the length length(a) of $a \in$ Cl that if $a \prec (x+1, \alpha)$, then there is an n so that $a \prec v(x+1, n)$. The proof depends on whether $\alpha = 0$ or $\alpha = \gamma+1$ or the successor of a successor. Exemplarily we treat the case $\alpha = 0$.
So assume $a \prec (x+1, 0)$. Either $a <' (x+1, 0)$ or $(x+1, 0) <' a$. If $(x+1, 0) <' a$, then $G(a) \leq |(x+1, 0)|$. Let $\gamma := G_v(x, 0)$. As $\omega^\omega \leq \gamma \in$ HZ, we have $\omega\gamma = \gamma$ and so $v(x, \omega\gamma) = v(x, \gamma)$. As $|(x, 0)| \leq G(x, 0) < G(x, \gamma)$, also $G(a) \leq |(x+1, 0)| < G(x, \gamma) = G_v(x, \gamma) = G_v(x+1, 0)$. That is, $a \prec v(x+1, 0)$. Next, we consider the case $a <' (x+1, 0)$. By I.H., $\text{eval}^{-1}(|a|) \prec v(x+1, n)$ for some n. Now for $\gamma := G_v(x+1, n)$, $a <' (x, \gamma)$ and $|a| < \gamma$, hence $G(a) < G(x, \gamma) = G_v(x, \gamma)$, so $G(a) < G_v(x, \gamma) \leq G_v(x+1, n+1)$.

(ii) This time, if $\alpha \in \mathsf{Lim}(\Omega)$, $\mathsf{sup}_{\xi<\omega(1+\alpha)} G_v(y[\xi], 0) = \mathsf{sup}_{\xi<\alpha} G_v(y[\omega(1+\xi)], 0) = \mathsf{sup}_{\xi<\alpha} G(y[\omega(1+\xi)], 0) = G(y, 0)$, and the claim is again by Lemma IV.4.7 and IV.4.9 and Lemma IV.2.7. Hence it remains to consider the case where $\omega(1+\alpha) = \delta+\omega$ for $\delta = 0$ or $\delta \in \mathsf{Lim}(\Omega)$. Again, we show by induction on the length length(a) of $a \in$ Cl that if $a \prec (y, 0)$, then there is an n so that $a \prec v(y[\delta+n], 0)$. If $(y, 0) <' a$, then $G(a) < |(y, 0)|$, thus $G(a) < G_v(y[\delta+n], \omega) < G_v(y[\delta+n+1], 0)$ for some n. And if $a <' (y, 0)$, then $a \prec (y, 0)$ implies that $G(a) < |(y, 0)|$, and $G(a) < G_v(y[\delta+n], \omega) < G_v(y[\delta+n+1], 0)$ for some n. That is, $a \prec v(y[\delta+n+1], 0)$.
□

Lemma IV.4.12. *Suppose that $y \in Q^H$ with $o(y) = \delta_0+\gamma_0$. Further, let $s_0 := G_v(y, \alpha)+1$, for $0 < \xi < \gamma_0$, $s_{\xi+1} := G_v(y[\delta_0+\xi], s_\xi)$ and for each $\gamma < \gamma_0$, $s_{\delta_0+\gamma} := G_v(y[\delta_0+\gamma], s_0)$. Then, for each $\gamma \leq \gamma_0$, $s_\gamma = \mathsf{sup}_{\xi<\gamma} s_\xi$.*

Proof We just consider the case $\delta_0 = 0$, the general case runs completely analogously. Let $\gamma' := G_v(y, \alpha) = G(y, \alpha)$. Then $s_0 := \gamma'+1$, hence for each $\xi < \gamma_0$, $G(y[\xi], \gamma') = \gamma'$ (cf. Lemma IV.2.7 (v)). Firstly, we show by induction on $\xi < \gamma$ that $G(y[\gamma], \alpha+1)$ is an upper bound of $(s_\xi : \xi \lhd \gamma)$, i.e. that for each $\xi < \gamma$, $s_\xi < G(y[\gamma], s_0)$. As $s_0 = \gamma'+1 < G(y[\gamma], \gamma'+1) = G(y[\gamma], s_0) \in E$, also $s_0+\omega < G(y[\gamma], s_0)$, and thus $s_1 = G_v(y[0], s_0) < G_v(y[0], s_0+\omega) < G(y[\gamma], s_0)$ by Lemma IV.4.8. If $0 < \xi < \gamma$, then s_ξ is a limit. By I.H., $s_\xi < G(y[\gamma], s_0)$, hence $s_{\xi+1} = G_v(y[\xi], s_\xi) \lhd (y[\gamma], s_0)$ readily follows.

Secondly, we show that $G(y[\gamma], s_0)$ is the least upper bound. By induction on the length length(a) of a, we show that if $a \prec v(y[\gamma], s_0) = (y[\gamma], s_0)$, then already $G(a) < s_\xi$ for some $\xi \lhd \gamma$. Thus assume $a \prec (y[\gamma], s_0)$. If $(y[\gamma], s_0) <' a$, then $G(a) < |(y[\gamma], s_0)| = s_0$ as $s_0 = G(y[\gamma_0], \alpha)+1$. And if $a <' (y[\gamma], s_0)$, then either $a = (y[\gamma], \beta)$ for $\beta < s_0$, and then $G(a) < s_0$, or $a = (z, \delta)$ for some z with $z < y[\gamma]$. As (by assumption) $G(a) < G(y[\gamma], s_0)$, $|a| < G(y[\gamma], s_0)$. By I.H. (on the length), there is a $\xi < \gamma$ so that $|a| < s_\xi$. We may assume that $z < y[\xi]$. Therefore, $G(a) < G_v(y[\xi], s_\xi) = s_{\xi+1}$. $\square$

Observe that in the above proof we employed transfinite induction up to γ_0. This is no problem, as the sequences $(s_\xi : \xi \lhd \gamma_0)$ play only a role in the proofs of Lemma III.7.10 (ii) and the corresponding results in Chapter II. There, we prove in ACA_0, that under some additional assumptions, if $x \in Q$, $\deg(x) = 1$ and $o(x) = \gamma_0$, then $\breve{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\mathcal{C}_x)$, where $\mathcal{C}_x := \{\alpha : \mathsf{Wo}_\lhd(g_{x^h}(\alpha))\}$.

Recall that Q also contains names of degree one the form $(\beta{+}1, (\gamma_0, z)^-) \circ y$. Then, $x^h = 1{+}x^H = (\beta{+}1, (\gamma_0, z^H)) \circ y^H$: the degree of x^h may increase, but still $o(x^h) = \gamma_0$ (cf. Definition III.4.20). By definition, $\breve{\mathsf{T}}_x$ implies $\mathsf{Wo}_{\rightsquigarrow^*}(x)$, which as $o(x) = \gamma_0$, entails $\mathsf{Wo}_\lhd(\gamma_0)$. Therefore, in the context where the above lemma is applied, $\mathsf{Wo}_\lhd(\gamma_0)$ is at hand. Also note that $\mathsf{Wo}_\lhd(\gamma_0)$ is already required to ensure that $(s_\xi : \xi \lhd \gamma_0)$ is total.

IV.5 Relating $\tilde{g}_x$ and the ϑ-function

In this section, we will see that the the OFG $\mathcal{O}_{H^*}$ defined in Lemma IV.2.23 is isomorphic to the OFG $\mathcal{O}_\vartheta = (\Sigma_{asc} \cup \varepsilon_{\Omega+1}, <_\vartheta, k_\vartheta, l_\vartheta)$, where $\varepsilon_{\Omega+1}$ is the first ε-number above Ω, and that $\mathrm{eval}_{\mathcal{O}_\vartheta} := \mathrm{eval}_\vartheta$ is the ϑ-function from [18], p.538, for the special case $|\mathfrak{X}| = \emptyset$.

We say that f is an isomorphism between the OFG $\mathcal{O}_1 := (A_1, <_1, k_1, l_1)$ and the OFG $\mathcal{O}_2 := (A_2, <_2, k_2, l_2)$, if $f : (A_1, <_1) \to (A_2, <_2)$ is an order-isomorphism, and for all $a \in A_1$, $k_1(a) = k_2(f(a))$ and $l_1(a) = l_2(f(a))$. It is immediate that then $C_{\mathcal{O}_1}(a) = C_{\mathcal{O}_2}(f(a))$, and therefore $\mathrm{eval}_{\mathcal{O}_1}(a) = \mathrm{eval}_{\mathcal{O}_2}(f(a))$.

Definition IV.5.1. *$\mathcal{O}_\vartheta = (A_\vartheta, <_\vartheta, k_\vartheta, l_\vartheta)$ is defined as follows.*

(i) $A_\vartheta := \Sigma_{asc} \cup \varepsilon_{\Omega+1}$,

(ii) $<_\vartheta \restriction \Sigma_{asc} :=<' \restriction \Sigma_{asc}$ *and* $<_\vartheta \restriction \varepsilon_{\Omega+1} = \in \restriction \varepsilon_{\Omega+1}$.

(iii) For $\alpha \in \varepsilon_{\Omega+1}$, $k_\vartheta(\alpha) := l_\vartheta(\alpha) := \{\alpha^*\}$, *and for* $\sigma \in \Sigma_{asc}$, $k_\vartheta(\sigma) := k(\sigma)$ *and* $l(\sigma) := l(\sigma)$, *where k and l are as in Definition IV.2.4. Further,* $|a|_\vartheta := \max(k_\vartheta(a) \cup \{-1\})$.

Recall that by definition of $\mathrm{eval}_\vartheta(a)$, we have that $\mathrm{eval}_\vartheta(a)$ is the least ξ so that $|a|_\vartheta \cup l_\vartheta(a) \subseteq \xi$, and for each $b \in A_\vartheta$, if $b <_\vartheta a$ and $|b|_\vartheta < \xi$, then $\mathrm{eval}_\vartheta(b) < \xi$.

Lemma IV.5.2. *For each $a \in \Sigma_{asc} \cup \varepsilon_{\Omega+1}$,*

$$eval_\vartheta(a) := \mathsf{min}\{\xi : \xi > |a|_\vartheta \cup l_\vartheta(a) \wedge (\forall b <_\vartheta a)(|b|_\vartheta < \xi \Rightarrow eval_\vartheta(b) < \xi)\}.$$

If $\sigma \in \Sigma_{asc}$ and $\alpha \in \varepsilon_{\Omega+1}$, then $\sigma <_\vartheta a$. Hence $(\forall b <_\vartheta \alpha)(|b|_\vartheta < \xi \Rightarrow \mathrm{eval}_\vartheta(b) < \xi)$ splits into the two clauses

(i) $(\forall \eta < \alpha)(\eta^* < \xi \Rightarrow \mathrm{eval}_\vartheta(\eta) < \xi)$, and

(ii) $(\forall \sigma \in \Sigma_{asc})(|\sigma|_\vartheta < \xi \Rightarrow \mathrm{eval}_\vartheta(\sigma) < \xi)$.

For $\sigma \in \Sigma_{asc}$, $\mathrm{eval}_\vartheta(\sigma) := \mathsf{min}\{\xi : \xi > |\sigma| \cup l(\sigma) \wedge (\forall \tau <_\vartheta \sigma)(|\tau| < \xi \Rightarrow \mathrm{eval}_\vartheta(\tau) < \xi)\}$. Therefore, an easy induction on $(\Sigma_{asc}, <_\vartheta)$ yields that $\mathrm{eval}_\vartheta(\sigma) = \mathrm{eval}_H(\sigma) = G(\sigma)$. Further, if $\alpha \in \varepsilon_{\Omega+1}$, then (ii) expresses that ξ is an ε-number. Hence, we have the following.

Lemma IV.5.3. *For each $\alpha \in \varepsilon_{\Omega+1}$,*

$$eval_\vartheta(\alpha) := \mathsf{min}\{\xi \in E : \xi > \alpha^* \wedge (\forall \eta < \alpha)(\eta^* < \xi \Rightarrow eval_\vartheta(\eta) < \xi)\}.$$

We use the ϑ-function from [18], p.538, for the special case where $|\mathfrak{X}| = \emptyset$.

Definition IV.5.4. *The sets of ordinals $C^n(\alpha, \beta)$, $C(\alpha, \beta)$ and the ordinals $\vartheta\alpha$ are defined for all $n \in \mathbb{N}$ and all ordinals α and β by recursion on α as follows.*

(i) $\{0, \Omega\} \cup \beta \subseteq C^n(\alpha, \beta)$,

(ii) *if* $\vec{\xi} \in C^n(\alpha, \beta)$ *and* $\xi =_{NF} \omega^{\xi_k} + \ldots + \omega^{\xi_0}$, *then* $\xi \in C^{n+1}(\alpha, \beta)$,

(iii) *if* $\xi \in C^n(\alpha, \beta) \cap \alpha$, *then* $\vartheta\xi \in C^{n+1}(\alpha, \beta)$,

(iv) $C(\alpha, \beta) = \bigcup_n C^n(\alpha, \beta)$,

(v) $\vartheta\alpha = \mathsf{min}\{\xi < \Omega : C(\alpha, \xi) \cap \Omega \subseteq \xi \wedge \alpha \in C(\alpha, \xi)\}$ *if there exists an ordinal* ξ *so that* $C(\alpha, \xi) \cap \Omega \subseteq \xi$ *and* $\alpha \in C(\alpha, \xi)$*; otherwise* $\vartheta\alpha$ *is undefined.*

The next lemma collect some standard properties of the ϑ-function. This lemma is taken form [18]. A proof can be found there.

Lemma IV.5.5. *$\vartheta\alpha$ is defined for each $\alpha < \varepsilon_{\Omega+1}$. Further, for all $\alpha, \beta < \varepsilon_{\Omega+1}$,*

(i) $\vartheta\alpha \in E$,

(ii) $\alpha \in C(\alpha, \vartheta\alpha)$,

(iii) $\vartheta\alpha = C(\alpha, \vartheta\alpha) \cap \Omega$, *and* $\vartheta\alpha \notin C(\alpha, \vartheta\alpha)$,

(iv) $\delta \in C(\alpha, \beta)$ *iff* $\delta^* \in C(\alpha, \beta)$,

(v) $\alpha^* < \vartheta\alpha$,

(vi) $\vartheta\alpha = \vartheta\beta$ *implies* $\alpha = \beta$,

(vii) $\vartheta\alpha < \vartheta\beta$ *iff* $(\alpha < \beta \wedge \alpha^* < \vartheta\beta) \vee (\beta < \alpha \wedge \vartheta\alpha \leq \beta^*)$.

(viii) $\beta < \vartheta\alpha$ *iff* $\omega^\beta < \vartheta\alpha$.

In order to relate ϑ to eval_ϑ, we slightly rewrite the definition of $\vartheta\alpha$.

Lemma IV.5.6. *For each* $\alpha < \varepsilon_{\Omega+1}$,

$$\vartheta\alpha = \xi_0 := \mathsf{min}\{\xi \in E : \xi > \alpha^* \wedge (\forall \eta < \alpha)(\eta^* < \xi \Rightarrow \vartheta\eta < \xi)\}.$$

Proof By definition, $\vartheta\alpha = \mathsf{min}\{\xi < \Omega : C(\alpha, \xi) \cap \Omega \subseteq \xi \ \wedge \ \alpha \in C(\alpha, \xi)\}$. Knowing that $\vartheta\alpha \in E$, it suffices to let range ξ over $E \cap \Omega$. As further, $\alpha \in C(\alpha, \xi)$ iff $\alpha^* \in C(\alpha, \xi) \cap \Omega$, we have that for each $\xi \in E \cap \Omega$,

$$C(\alpha, \xi) \cap \Omega \subseteq \xi \Leftrightarrow (\forall \eta < \alpha)(\eta^* < \xi \Rightarrow \vartheta\eta < \xi).$$

Moreover, given $C(\alpha, \xi) \cap \Omega \subseteq \xi$, $\alpha \in C(\alpha, \xi)$ iff iff $\alpha^* < \xi$. Thus the claim follows. □

Analogously to Lemma IV.2.12, we obtain by the following.

Lemma IV.5.7.

(i) For each $a \in A_\vartheta \cap \Sigma_{asc}$, $\mathit{eval}_\vartheta(a) = \mathit{eval}_H(a) = G(a)$.

(ii) For each $\alpha \in A_\vartheta \cap \varepsilon_{\Omega+1}$, $\mathit{eval}_\vartheta(\alpha) = \vartheta\alpha$.

Proof (i) See the discussion following Lemma IV.5.2. (ii) Let $\alpha \in A_\vartheta$ and suppose that the claim holds for each $\beta <_\vartheta \alpha$ (i.e. $\beta \in \alpha$) with $\beta \in A_\vartheta$. By the above Lemma and Lemma IV.5.3, we obtain that for each $\alpha \in A_\vartheta \cap \varepsilon_{\Omega+1}$,

$$\begin{aligned}\mathrm{eval}_\vartheta(\alpha) = \mathsf{min}\{\xi \in E : \xi > \alpha^* \ \wedge \ (\forall \beta <_\vartheta \alpha)(\beta^* < \xi \Rightarrow \mathrm{eval}_\vartheta(\beta) < \xi)\} =_{IH} \\ \mathsf{min}\{\xi \in E : \xi > \alpha^* \ \wedge \ (\forall \beta <_\vartheta \alpha)(\beta^* < \xi \Rightarrow \vartheta\beta < \xi)\} = \vartheta\alpha.\end{aligned}$$

□

Hence we have that $\mathrm{eval}_\vartheta(\alpha) = \vartheta\alpha$. To see that also $\mathrm{eval}_{H^*}(\alpha) = \vartheta\alpha$, we just have to show that the OFG $\mathcal{O}_{H^*}$ is isomorphic to the OFG $\mathcal{O}_\vartheta$. We start by defining an order-isomorphism between $(Q^H, <)$ and $(\varepsilon_{\Omega+1}, \in)$.

Definition IV.5.8. *We assign to each $x \in Q^H$ an ordinal $\|x\|$ below $\varepsilon_{\Omega+1}$ as follows:*

(i) $\|q_0\| := 0$,

(ii) $\|\langle(\alpha_k, x_k), \ldots, (\alpha_1, x_1)\rangle\| := \Omega^{\|x_1\|}\alpha_1 + \ldots + \Omega^{\|x_k\|}\alpha_k$.

As we work with $\tilde{Q}$ rather than Q^H, we further need an order-isomorphism between $(Q^H, <)$ and $(\tilde{Q}, <)$. To define it, let $L := \{x \in Q^H : \deg(x) > 1 \wedge o(x) = 1\}$, so each $x \in L$ is a limit in $(Q^H, <)$. It is readily seen that $\oplus_L : (Q^H, <) \to (\tilde{Q}_0, <)$ is an order-isomorphism. An order-isomorphism between $(Q^H, <)$ and $(\tilde{Q}, <)$ is then given by the function which maps $1{+}x$ to $x \oplus_L 1$, where for $x \in Q^H$, $1{+}x := x{+}1$ if $x < (\omega, q_0)$, and $1{+}x := x$ if $x \geq (\omega, q_0)$.

We also give the reverse direction of the isomorphism that maps $1{+}x$ to $x \oplus_L 1$, in order to define an order-isomorphism between $(\tilde{Q} \times \Omega, <')$ and $(\varepsilon_{\Omega+1}, \in)$.

Definition IV.5.9. *Let $L := \{x \in Q^H : \deg(x) > 1 \wedge o(x) = 1\}$. Then $i_0 : \tilde{Q}_0 \to Q^H$ and $i_1 : \tilde{Q} \to Q^H$ are defined as follows. If $x = y{+}n{+}1$ for some $y \in L$, then $i_0(x) := y{+}n$, else $i_0(x) := x$. And $i_1(x) := i_0(x)$ if $x \geq (\omega, q_0)$, and $i_1(n{+}1, q_0) := i_0(n, q_0)$.*

An order-isomorphism between $\tilde{Q} \times \Omega$ and $(\varepsilon_{\Omega+1}, \in)$ is then provided by the map $\|x\|' := \|i_1(x)\|$, and an order-isomorphism between $\tilde{Q} \times \Omega$ and $(\varepsilon_{\Omega+1}, \in)$ is then provided by the map $(x, \alpha) \mapsto \Omega\|x\|'{+}\alpha$.

Lemma IV.5.10. *We have the following:*

(i) $(\alpha{+}\beta)^* = \mathsf{max}(\alpha^*, \beta^*)$, $\Omega^* = 0$ *and* $(\Omega{+}\alpha)^* = \alpha^*$.

(ii) $(\Omega^{\alpha_1}\beta_1{+}\ldots{+}\Omega^{\alpha_k}\beta_k)^* = \mathsf{max}\{\alpha_1^*, \ldots, \alpha_k^*, \beta_1^*, \ldots, \beta_k^*\}$.

(iii) For each $x \in \tilde{Q}$, $\|x\|^* = |x|^* = (|x|')^*$.

Proof (i) is immediate by the definition of $\cdot^*$. (ii) Since $\Omega\alpha = \omega^\Omega\alpha$, $(\Omega\alpha)^* = \alpha^*$. So, as $\Omega^\alpha\beta = \omega^{\Omega\alpha+\beta}$, $(\Omega\alpha + \beta)^* = \mathsf{max}(\alpha^*, \beta^*)$ by (i). Now (ii) follows from (i).
(iii) As $|b \oplus 1| \leq |b|{+}1$ and $\alpha^* = (\alpha{+}1)^*$, we just have to show the first equality, which is done by induction on the build-up of $x := \langle(\beta_1, x_1), \ldots, (\beta_k, x_k)\rangle \in \tilde{Q}$:

$$\|x\|^* = (\Omega^{\|x_1\|}\beta_1{+}\ldots{+}\Omega^{\|x_k\|}\beta_k)^* = \mathsf{max}(\beta_1^*, \ldots, \beta_k^*, \|x_1\|^*, \ldots, \|x_k\|^*) =_{IH}$$
$$\mathsf{max}(\beta_1^*, \ldots, \beta_k^*, |x_1|^*, \ldots, |x_k|^*) = |x|^*.$$

□

Finally, we can define the sought-after isomorphism $t : \mathcal{O}_{H^*} \to \mathcal{O}_\vartheta$.

Lemma IV.5.11. *Let $t : \Sigma_{asc} \cup \tilde{Q} \times \Omega \to A_\vartheta$ be as follows:*

$$t(a) := \begin{cases} a & : a \in \Sigma_{asc}, \\ \Omega\|x\|'{+}\alpha : & a = (x, \alpha) \in \tilde{Q} \times \Omega. \end{cases}$$

Then, $t : \mathcal{O}_{H^*} \to \mathcal{O}_\vartheta$ is an isomorphism between OFGs.

Proof Since $t : \Sigma_{asc} \cup \tilde{Q} \times \Omega \to A_\vartheta$ is an order-isomorphism, it remains to check that for each $a \in \Sigma_{asc} \cup \tilde{Q} \times \Omega$, $k^*(a) = k_\vartheta(t(a))$ and $l^*(a) = l_\vartheta(t(a))$. However, this is by Lemma IV.5.10 (iii). □

As further, by Lemma IV.2.23, $\text{eval}_{H'} = \text{eval}_{H^*}$ and by Lemma IV.2.17, for $\oplus := \oplus_{(\tilde{Q}\times\Omega)\setminus\text{NF}_H}$ and each $a \in A[\text{Cl}_H]$, $\text{eval}_H(a \oplus 1) = \text{eval}_{H'}(a)$, we have the following.

Theorem IV.5.12. *If $x \in \tilde{Q}$ and $\gamma, |x| < \varepsilon_{\Omega+1}$, then $eval_H((x,\alpha){\oplus}1) = \vartheta(\Omega\|x\|'{+}\alpha)$.*

All we actually use of this theorem are some the following instances. Recall that $g_x := H_x(g)$ for $g(\alpha) := \omega^{1+\alpha}$. Below, $q_2^\omega := (1, q_1^\omega) = (1, (\omega, q_0))$ is the name of the functional $\mathsf{It}^\omega(\mathsf{it})$.

Corollary IV.5.13. *Let $x \in Q^H$. If $\deg(x) > 1$, $x \geq q_2^\omega$ and $|x| \leq \varepsilon_0$, then we have the following:*

(i) $g_{x+1}(\omega) = \vartheta\|x\|$,

(ii) *if $o(x) = 1$, then $g_x(\omega) = \vartheta\|x[\omega]\|$ and $g_x(\varepsilon_0) = \vartheta\|x[\varepsilon_0]\|$.*

Proof Since $|(x, \alpha)| \leq \varepsilon_0$, we have that $(x, \alpha) \in \tilde{Q} \times \Omega$ implies that $(x, \alpha) \in \text{NF}_H$, and so $(x, \alpha) \oplus_L 1 = (x, \alpha)$ for $L := (\tilde{Q} \times \Omega) \setminus \text{NF}_H$. Further, if $\deg(x) > 1$ and $x > q_2^\omega$, then $x[\omega] \geq q_2^\omega$, and if $x \geq q_2^\omega$, then $\|x\| \geq \Omega^\omega$, and thus $\|x\| = \Omega\|x\|$. Now both claims easily follow:
(i) $g_{x+1}(\omega) = \tilde{g}_{x+1}(0) = \text{eval}_H(x{+}1, 0) = \text{eval}_H((x{+}1, 0) \oplus_L 1)$. Since $\deg(x) > 1$, $\|x{+}1\|' = \|x\|$ by definition of $\|\cdot\|'$, hence the claim. (ii) As $o(x) = 1$ and $\gamma \leq \varepsilon_0$, then $g_x(\gamma) = g_{x[\gamma]}(0) = \tilde{g}_{x[\gamma]}(0) = \text{eval}_H((x[\gamma], 0) \oplus_L 1)$. By definition of $\|\cdot\|'$, $\|x[\gamma]\|' = \|x[\gamma]\|$. The claim follows. □

IV.5.1 The *n*ary Veblen functions and $(\tilde{g}_x : x \in \tilde{Q}_0 \wedge x < q_2^\omega)$

Since $q_2^\omega := (1, q_1^\omega) = q_3[\omega]$, we have by Corollary IV.5.13 that $\tilde{g}_x(0) = g_x(0) = \mathsf{It}^\omega[\mathsf{it}, g, 0] = \vartheta\Omega^\omega$ is the small Veblen number. So, the functions $(\tilde{g}_x : x \in \tilde{Q}_0 \wedge x < q_2^\omega)$ relate to the *n*ary Veblen functions: for $L := \{x \in Q^H : \deg(x) > 1 \wedge o(x) = 1\}$ and $x \in Q^H$ with $x < q_2^\omega$, we have that $\tilde{g}_{x\oplus 1} = \varphi_x$, as we will show below (cf. Lemma IV.5.15).

The following definition of the Veblen functions is taken from Setzer [24].

Definition IV.5.14. *[The k+2-ary Veblen function]*
$\varphi^{k+2} : \Omega^{k+2} \to \Omega$ is defined by recursion on the lexicographic ordering on Ω^{k+2}.

(i) $\varphi^{k+2}(0,\dots,0,\alpha_0) := \omega^{1+\alpha_0}$*, and if* $\alpha_{i+1} \neq 0$*, then*

(ii) $\varphi^{k+2}(\alpha_{k+1},\dots,\alpha_{i+1},0,\dots,0,\alpha_0)$ *is the* α_0*th common fixed point of the functions* $h_\alpha : \Omega \to \Omega$*,* $(\alpha < \alpha_{i+1})$ *with* $h_\alpha(\xi) := \varphi^{k+2}(\alpha_{k+1},\dots,\alpha_{i+2},\alpha,\xi,0,\dots,0)$.

A name $x \in Q^H$ with $x < q_2^\omega$ has to form $x = \langle(\beta, q_0)\rangle * \langle(\alpha_{n_1}, q_1^{n_1}), \dots (\alpha_{n_k}, q_1^{n_k})\rangle$, with $n_1 < \dots < n_k$, where β may be 0 (in which case $(x)_0 = (\alpha_{n_1}, q_1^{n_1})$, as we write $(0, q_0)$ for $\langle\rangle$). For the following, it is convenient to identify a k-tuple $(\alpha_k,\dots,\alpha_1)$ with the name $\langle\alpha_1, q_1^0\rangle * \langle\alpha_2, q_1^1\rangle * \dots * \langle\alpha_k, q_1^{k-1}\rangle$, where $q_1^0 = q_0$. Further, we write $\varphi_{(\alpha_k,\dots,\alpha_1)}$ for the function $\alpha_0 \mapsto \varphi(\alpha_k,\dots,\alpha_0)$.

In order to formulate the next lemma, we let $L := \{x \in Q^H : \deg(x) > 1 \wedge o(x) = 1\}$ and $\oplus := \oplus_L$. Then, $\oplus : Q^H \to \tilde{Q}_0$, and $\tilde{g}_{x\oplus 1}$ is defined for all $x \in Q^H$.

Lemma IV.5.15. *For each $x \in Q^H$ with $x < q_2^\omega$, $\tilde{g}_{x\oplus 1} = \varphi_x$.*

Proof By induction on $(Q^H, <)$. Recall that $\mathsf{sh} \circ \mathsf{fix} = \mathsf{fix}$ (cf. Lemma I.3.15) and thus $\tilde{g}'_x = g'_x$. By definition, $\varphi^2_{(0)} = g = \tilde{g}_{q_0}$. If $x \neq q_0$, we consider the following two cases. Firstly, let $y := (\alpha_{k+1},\dots,\alpha_1)$ and $x := y{+}1$ (so $x = (\alpha_{k+1},\dots,\alpha_1{+}1)$). Then, $\varphi_x^{k+2} = \mathsf{fix}(\xi \mapsto \varphi_y^{k+2}(\xi)) =_{IH} g'_{y\oplus 1} = \tilde{g}_{x\oplus 1}$. Secondly, let $x := (\alpha_{k+1},\dots,\alpha_{i+2}{+}1,\vec{0})$ (that is, $\alpha_{i+1} = \dots = \alpha_1 = 0$). Then $x[\xi] = (\alpha_{k+1},\dots,\alpha_{i+2},1{+}\xi,\vec{0})$, and

$$\varphi_x^{k+2} = \mathsf{fix}(\xi \mapsto \varphi^{k+2}(\alpha_{k+1},\dots,\alpha_{i+2},1{+}\xi,\vec{0})) =_{IH}$$
$$\mathsf{fix}(\xi \mapsto \tilde{g}_{x[\xi]\oplus 1}(0)) = \mathsf{fix}(\xi \mapsto g_{x[\xi]}(0)) = g'_x = \tilde{g}_{x+1} = \tilde{g}_{x\oplus 1}.$$

Here, we used that $\xi \mapsto \tilde{g}_{x[\xi]\oplus 1}$ and $\xi \mapsto g_{x[\xi]}$ have the same fixed points. As only limit ordinals are fixed point candidates, and $\tilde{g}_{x[\gamma]\oplus 1} = g_{x[\gamma]}$ by definition, this clearly is the case. Finally, let $x = (\alpha_{k+1},\dots,\alpha_{i+2},\gamma,\vec{0})$. Note that for $\alpha_{i+1} < \gamma$, $x[\alpha_{i+1}] = (\alpha_{k+1},\dots,\alpha_{i+2},1{+}\alpha_{i+1},\vec{0})$. This time, we have

$$\varphi_x^{k+2} = \bigcap_{\alpha_{i+1}<\gamma} \mathsf{fix}(\xi \mapsto \varphi^{k+2}(\alpha_{k+1},\dots,\alpha_{i+2},\alpha_{i+1},\xi,\vec{0})) =$$
$$\bigcap_{\alpha_{i+1}<\gamma} \mathsf{fix}(\xi \mapsto \varphi^{k+2}(\alpha_{k+1},\dots,\alpha_{i+1},1{+}\xi,\vec{0})) =_{IH}$$
$$\bigcap_{\alpha_{i+1}<\gamma} g'_{(\alpha_{k+1},\dots,\alpha_{i+2},\alpha_{i+1}+1,\vec{0})\oplus 1} =_{L.III.5.3} g_{(\alpha_{k+1},\dots,\alpha_{i+2},\gamma,\vec{0})} = g_x = \tilde{g}_{x\oplus 1}.$$

□

All we actually use of this lemma are some the following instances. Below, we write $\varphi\alpha_{k+1}\dots\alpha_0$ for $\varphi^{k+2}(\alpha_{k+1}\dots\alpha_0)$.

Corollary IV.5.16. *Let $x \in Q^H$. We have the following.*

(i) $g_{(n+1,q_0)}(\omega) = \varphi(n+1)0$ *and* $g_{(n+1,q_0)}(\varepsilon_0) = \varphi(n+1)\varepsilon_0$*,*

(ii) if $\deg(x) > 1$ *and* $o(x) = 1$*, then* $g_{x[\gamma]} = \varphi_{x[\gamma]}$*.*

(iii) if $\deg(x) > 1$ *and* $o(x) = 1$*, then* $g_{x+1}(\omega) = \varphi_x 0$ *and* $g_{x+1}(\varepsilon_0) = \varphi_x \varepsilon_0$*.*

Proof Let $L := \{x \in Q^H : \deg(x) > 1 \wedge o(x) = 1\}$. Note that $(n+1, q_0) \oplus 1 = (n+1, q_0)$, and if $\deg(x) > 1$ and $o(x) = 1$, then $x \oplus_L 1 = x+1$ and $x[\gamma] \oplus_L 1 = x[\gamma]$. Further observe that for each $x \in Q^H$, $g_{x+1}(\omega) = \tilde{g}_{x+1}(0)$ and $g_{x+1}(\varepsilon_0) = \tilde{g}_{x+1}(\varepsilon_0)$, and that $g_{x[\gamma]} = \tilde{g}_{x[\gamma]}$. The claim is now by the above lemma. $\square$

Part II

Bounds

Chapter V

The infinitary systems $(\overset{*}{\mathsf{T}}_x : x \in Q)$

In this chapter, we introduce most of the notions used to compute bounds of the theories $(\mathsf{T}_x : x \in Q)$. For reasons discussed below, we work in a language $\mathcal{L}$ which extends L_2 by additional relation symbols $\mathsf{U}_1, \mathsf{U}_2, \ldots$ And since we are dealing with cut-elimination, we introduce for each of the theories $(\mathsf{T}_x : x \in Q)$ a finitary and an infinitary Tait-style system.

The notion of a bound $f : \Omega \to \Omega$ of T_x is tied to the infinitary Tait-style system $\overset{*}{\mathsf{T}}_x$ corresponding to the theory T_x, and ceils the costs of eliminating cuts in the following way: for each limit ordinal γ and each finite set Γ of arithmetical formulas,

$$\overset{*}{\mathsf{T}}_x \mathrel{\Big|\!\frac{<\gamma}{+}} \Gamma \Longrightarrow \overset{*}{\mathsf{T}}_x \mathrel{\Big|\!\frac{<f(\gamma)}{-}} \Gamma,$$

where in the derivation on the left, the cut-rule is restricted to instances of axioms of $\overset{*}{\mathsf{T}}_x$ and some further formulas that do not impede the cut-elimination process, while the derivation on the right is cut-free. Since Γ is arithmetical, we also have $\overset{*}{\mathsf{T}}{}^{\epsilon} \mathrel{\Big|\!\frac{<f(\gamma)}{-}} \Gamma$.

We will see that the function g_{x^h}, which we have shown to be provable in T_x in the first part, is also a bound of T_x. Moreover, if h is provable in T_x and f is a bound of T_x, then $h{\restriction}\mathsf{Lim}(\Omega) \leq f{\restriction}\mathsf{Lim}(\Omega)$. Therefore, g_{x^h} is the largest normal function that is provable in T_x, and at the same time, the least bound of T_x (in the sense that if f is another bound, then $g_{x^h}{\restriction}\mathsf{Lim}(\Omega) \leq f{\restriction}\mathsf{Lim}(\Omega)$). We call a normal function which is provable in T_x, and at the same time a bound of T_x, a sharp bound of T_x.

In order to deal with the operation p_1, we also stack theories on top of each other: $\mathsf{T}_1|\mathsf{T}_0$ ("T_1 over T_0") is essentially the theory $\mathsf{T}^{\epsilon} + \breve{\mathsf{T}}_1 \wedge \exists X(\breve{\mathsf{T}}_0{\restriction}X)$, that is, $\mathsf{T}_1|\mathsf{T}_0$ extends T_1 by an axiom asserting that there is an ω-model of T_0. For the subsequent arguments it proves however more convenient to have an explicit class term for the ω-model above T_0, say $\{x : \mathsf{U}_1(x)\}$, where U_1 is a fresh relation symbol. That is why we work in this second part mainly in the languages $\mathcal{L}$ which extends L_2 by additional relation symbols $\mathsf{U}_i(u)$ for each $0 < i$, where $\mathsf{U}_0 := \mathsf{U}$.

The motivation for stacking theories on top of each other is that the theory $\mathsf{p}_1(\mathsf{T})$ asserts that above each set Z there is a set X with $Z \dot{\in} X$ and $\check{\mathsf{T}}{\restriction}X$. As a proof in $\mathsf{p}_1(\mathsf{T})$ makes only use of finitely many instances of this assertion, it seems at least plausible that for each arithmetical L_2-formula A with $\mathsf{FV}_1(A) = \emptyset$, $\mathsf{p}_1(\mathsf{T}) \vdash A$ entails that for some $n \in \mathbb{N}$, A is already provable in T^ϵ from the assumption that there are sets $X_0 \dot{\in} \ldots \dot{\in} X_n$ with $\check{\mathsf{T}}{\restriction}X_i$ for each $i \leq n$, and this assumption is provided by the theory $\underbrace{\mathsf{T}|\ldots|\mathsf{T}}_{n}$.

We start this chapter by having a closer look at theories of the form $\mathsf{T}_1|\mathsf{T}_0$ (we call $\mathsf{T}_1|\mathsf{T}_0$ also the composition of T_0 with T_1, since if f_0 is a sharp bound of T_0 and f_1 is a sharp bound of T_1, then $f_0 \circ f_1$ is a sharp bound of $\mathsf{T}_1|\mathsf{T}_0$).

V.1 The language $\mathcal{L}$ and composition of theories

In this second part, we work with the languages $\mathcal{L}$ which extends L_2 by additional relation symbols $\mathsf{U}_i(u)$ for each $0 < i$, where $\mathsf{U}_0 := \mathsf{U}$. Again, U_i denotes also the class term $\{x : \mathsf{U}_i(x)\}$. It is assumed that the free number variables of the language $\mathcal{L}$ are $u_0, u_1, \ldots$, and the free number variables are $U_0, U_1, \ldots$ Moreover, we extend the theory T^ϵ to the language $\mathcal{L}$, which in particular means that we have an axiom $\mathsf{U}_i(u) \vee \neg\mathsf{U}_i(u)$ for each $i \in \mathbb{N}$.

In the sequel, we often care which relation symbols U_i occur in a formula A. The relation symbols U_i are used to axiomatize theories of the form $\mathsf{T}_k|\ldots|\mathsf{T}_0$ (see Definition V.1.6). We let S range over such theories which are also of the form $\mathsf{T}^\epsilon + \check{\mathsf{S}}$.

Definition V.1.1. *We write $A \in \mathcal{L}$ to indicate that A is an $\mathcal{L}$-formula. If an $\mathcal{L}$-theory S is given, then we say that $A \in \mathcal{L}(\mathsf{S})$, if A contains besides the relation symbol U at most the relation symbols U_i that occur in $\check{\mathsf{S}}$. The set of $\mathcal{L}(\mathsf{S})$-literals is denoted by $\mathcal{L}_{lit}(\mathsf{S})$, and the set of arithmetical $\mathcal{L}(\mathsf{S})$-formulas is denoted by $\mathcal{L}_{\Pi^1_0}(\mathsf{S})$.*

It also proves convenient to consider the sets $\mathcal{L}_{\mathsf{e}\Pi^1_n}$ and $\mathcal{L}_{\mathsf{e}\Sigma^1_n}$ of formulas that are essentially Π^1_n and essentially Σ^1_n defined below.

Definition V.1.2. *Let $\mathbb{C}$ be a set of $\mathcal{L}$-formulas. Then, the set $\mathsf{e}\Sigma(\mathbb{C})$ is the smallest superset of $\mathbb{C}$ that is closed under conjunction, disjunction and existential quantification in both sorts. Accordingly, $\mathsf{e}\Pi(\mathbb{C})$ is the smallest superset of $\mathbb{C}$ that is closed under conjunction, disjunction, and universal quantification in both sorts. Further, $\Sigma(\mathbb{C}) := \{\exists X A : A \in \mathbb{C}'\}$, where $\mathbb{C}'$ is $\mathbb{C}$ closed under conjunction and disjunction, and $\Pi(\mathbb{C}) := \{\forall X A : A \in \mathbb{C}'\}$.*

Note that $A \in \mathsf{e}\Pi(\mathbb{C})$ iff $\neg A \in \mathsf{e}\Sigma(\neg\mathbb{C})$.

Definition V.1.3. *$\mathcal{L}_{\mathsf{e}\Sigma^1_0} := \mathcal{L}_{\mathsf{e}\Pi^1_0} := \mathcal{L}_{\Pi^1_0}$, and $\mathcal{L}_{\mathsf{e}\Sigma^1_{n+1}} := \mathsf{e}\Sigma(\mathcal{L}_{\mathsf{e}\Pi^1_n})$ and $\mathcal{L}_{\mathsf{e}\Pi^1_{n+1}} := \mathsf{e}\Pi(\mathcal{L}_{\mathsf{e}\Sigma^1_n})$.*

It is obvious that over a theory T that implies arithmetical comprehension, each formula in $\mathcal{L}_{\mathsf{e}\Sigma^1_n}$ is equivalent to one in $\mathcal{L}_{\Sigma^1_n}$ with the same free variables. An analogous result holds for the formulas in $\mathcal{L}_{\mathsf{e}\Pi^1_n}$.

Lemma V.1.4. *Let $A \in \mathcal{L}_{\mathsf{e}\Sigma^1_n}$. If S comprises (ACA), then there is an $A' \in \mathcal{L}_{\Sigma^1_n}$ with $\mathsf{FV}(A) = \mathsf{FV}(A')$ so that $\mathsf{S} \vdash A \leftrightarrow A'$.*

Remark V.1.5. *Recall that in the first part, we said that A is Π^1_2, if $\mathsf{T}^\epsilon \vdash A \leftrightarrow A'$ for some Π^1_2-formula A' with $\mathsf{FV}(A) = \mathsf{FV}(A')$. As it is assumed that each theory T implies arithmetical comprehension, we have e.g. that $\breve{\mathsf{T}} \in \mathcal{L}_{\mathsf{e}\Pi^1_n}$ implies that $\breve{\mathsf{T}}$ is Π^1_n in the above sense. However, note that A is Π^1_n does not imply $A \in \mathcal{L}_{\mathsf{e}\Pi^1_n}$: for instance, let $A := \mathsf{p}_2((\mathsf{ACA})) \wedge \forall x \exists X \forall y \exists Y B$ for an arithmetical B. Then, A is only in $\mathcal{L}_{\mathsf{e}\Pi^1_4}$, but A is Π^1_3, as over $\mathsf{p}_2(\mathsf{ACA}_0)$ (already Σ^1_1-AC_0), $\forall x \exists X \forall y \exists Y B$ is equivalent to a Σ^1_1-formula.*

The purpose of the additional relation symbols in the language $\mathcal{L}$ is to have a specific set term that denotes an ω-model of the theory S when we have a look at the composition of S with T, or as we usually put it, the theory $\mathsf{T}|\mathsf{S}$ (”T” over ”S”) defined below.

Definition V.1.6. *Assume that $\mathsf{T}, \mathsf{T}_0, \mathsf{T}_1, \ldots$ are L_2-theories that imply (ACA). Then,*

(i) $\mathsf{T}|\mathsf{T}^\epsilon := \mathsf{T}$, *and* $\mathsf{T}^\epsilon|\mathsf{T}^\epsilon := \mathsf{T}^\epsilon$.

(ii) *if* $\mathsf{S} := \mathsf{T}_k|\ldots|\mathsf{T}_0$, *then* $\mathsf{T}_{k+1}|\mathsf{S} := \mathsf{T}_{k+1} + \breve{\mathsf{S}}{\restriction}\mathsf{U}_{k+1} \wedge \exists X[X = \mathsf{U}_{k+1}]$, *and*
$\mathsf{T}^\epsilon|\mathsf{S} := \mathsf{T}^\epsilon + \breve{\mathsf{S}}{\restriction}\mathsf{U}_{k+1}$, *where* $\breve{\mathsf{S}}$ *is the axiom of* S *besides the axioms of* T^ϵ.

Further, $\mathsf{T}^0 := \mathsf{T}^\epsilon$ and $\mathsf{T}^{n+1} := \mathsf{T}|\mathsf{T}^n$. Moreover, $\breve{\mathsf{T}}|\mathsf{S}$ denotes the axiom of $\mathsf{T}|\mathsf{S}$ besides the axioms of T^ϵ.

By induction on k it is readily observed that the theory $\mathsf{T}_{k+1}|\ldots|\mathsf{T}_0$ proves that $\mathsf{U}_0 \dot{\in} \ldots \dot{\in} \mathsf{U}_{k+1}$, that U_i is a set for each $i \leq k{+}1$, and that $\breve{\mathsf{T}}_i{\restriction}\mathsf{U}_{i+1}$ for each $i \leq k$. This is the case as each T_i implies (ACA) and thus $(\mathsf{ACA}){\restriction}\mathsf{U}_{i+1}$ for each $i \leq k$.

Convention V.1.7. *As before, $\mathsf{T}, \mathsf{T}', \mathsf{T}_0, \mathsf{T}_1, \ldots$ range over L_2-theories that imply (ACA). Further, we let S range over $\mathcal{L}$-theories of the form $\mathsf{T}_k|\ldots|\mathsf{T}_0$ or T or T^ϵ. If we introduce a theory as $\mathsf{T}_k|\ldots|\mathsf{T}_0|\mathsf{S}$, then for $0 \leq n \leq k$, M_n refers to U_n in case that S is T^ϵ, and in case that S is $\mathsf{T}_{k'}|\ldots|\mathsf{T}_0$, then M_n refers to $\mathsf{U}_{(k'+1)+n}$; in particular, M_0 refers to $\mathsf{U}_{k'+1}$ an is thus an ω-model of S.*

The theories T that we have encountered in the first part, say ACA_0 and $\mathsf{Op}_x(\mathsf{ACA}_0)$ for $x \in Q^*$, have all the property that with $\mathsf{T} \vdash A(\mathsf{U})$ also $\mathsf{T} \vdash \forall X A(X)$, in particular, $\mathsf{T} \vdash \mathsf{TI}_\lhd(\mathsf{U}, \alpha)$ iff $\mathsf{T} \vdash \mathsf{Wo}_\lhd(\alpha)$. This is the case as all a theory T claims about U is that it is a set, and further, the operations Op_x do not treat any set special.

As theories of the form $\mathsf{T}|\mathsf{T}'$ lack this property, we adjust the notion of a provable function (cf. Definitions III.7.3 and III.7.2). To do so, we introduce composite names to address theories of the form $\mathsf{S} := \mathsf{T}_k \ldots |\mathsf{T}_0$, and we assign to each composite name c a degree and a class term $\mathcal{C}^c$ (cf. Definition III.7.2).

Definition V.1.8. *A composite name c of length n is an element of Q^n (an n-tuple $(x_1, \ldots, x_k)$ of names $x_1, \ldots, x_k \in Q$). The empty composite name is denoted by $\Box$ and $\mathsf{T}^\Box := \mathsf{T}^\epsilon$. And for $c := (x_1, \ldots, x_k)$, we let*

(i) $\mathsf{T}^c := \mathsf{T}_{x_1}| \ldots |\mathsf{T}_{x_k}$, *and* $H^c := H_{x_k} \circ \ldots \circ H_{x_1}$,

(ii) $\deg(c) := \deg(x_1)$, $\mathcal{C}^c := \mathcal{C}_{x_1}$ *and* $c^h := (x_1^h, \ldots, x_k^h)$,

(iii) $\mathsf{Prv}_0(c) := \breve{\mathsf{T}}^c \to \forall\alpha[\mathsf{TI}_\lhd(\mathcal{C}^c, \alpha) \wedge \mathsf{Wo}_\lhd(\alpha) \to \mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(\alpha))]$, *where* $g(\alpha) := \omega^{1+\alpha}$ *and* $g^{c^h} := H^{c^h}(g)$. *We say that* T^c *proves* g^{c^h}, *if* $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(c)$.

If $x_0, \ldots, x_k \in Q$, then $(x_0, \Box) := x_0$ and $(x_0, (x_1, \ldots, x_k)) := (x_0, x_1, \ldots, x_k)$.

We conclude this section by showing that for each composite name c, T^c proves g^{c^h}. To show this, we use that if $\mathsf{S} \vdash A$, then $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \models A$, which is rather obvious. We give a proof, though, but only after we have introduced Tait-style systems (cf. Lemma V.2.8 (i)).

Lemma V.1.9. *Let $\mathsf{T} := \mathsf{ACA}_0$, $g(\alpha) := \omega^{1+\alpha}$ and c a non-empty composite name. Then T^c proves g^{c^h}.*

Proof Since the composite name c is represented by a closed term, we have that $\mathsf{T}^\epsilon \vdash \mathsf{Prv}_0(c)$ iff $\mathsf{T}^c \vdash \mathsf{TI}_\lhd(\mathcal{C}^c, \alpha) \wedge \mathsf{Wo}_\lhd(\alpha) \to \mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(\alpha))$.
We prove the claim by induction on the length of c. If $c = (x_0)$ then the claim is by Corollary III.7.16. Next, we assume that $c := (x_1, \ldots, x_k)$ is a non-empty composite name so that T^c proves g^{c^h} and that $x_0 \in Q$. We have to show $\mathsf{T}_{x_0}|\mathsf{T}^c \vdash \mathsf{Prv}_0((x_0, c))$. Thereto, we work informally in $\mathsf{T}_{x_0}|\mathsf{T}^c$. To show that $\mathsf{Prv}_0((x_0, c))$, assume that $\mathsf{TI}_\lhd(\mathcal{C}_{x_0}, \alpha) \wedge \mathsf{Wo}_\lhd(\alpha)$, and aim for $\mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(g_{x_0^h}(\alpha)))$. By Corollary III.7.16, T_{x_0} proves $g_{x_0^h}$, therefore we obtain $\mathsf{Wo}_\lhd(g_{x_0^h}(\alpha))$. As T^c proves g^{c^h} by assumption, we have

$$\mathsf{T}^c \vdash \forall\beta[\mathsf{TI}_\lhd(\mathcal{C}_{x_1}, \beta) \wedge \mathsf{Wo}_\lhd(\beta) \to \mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(\beta))].$$

By the above remark (or Lemma V.2.8 (i)), we have for $\mathsf{M}_0 := \mathsf{U}_k$,

$$(*) \qquad \mathsf{T}_{x_0}|\mathsf{T}^c \vdash \forall\beta[\mathsf{TI}_\lhd(\mathcal{C}^{\mathsf{M}_0}_{x_1},\beta) \wedge \mathsf{Wo}^{\mathsf{M}_0}_\lhd(\beta) \to \mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(\beta))].$$

As $\mathcal{C}^{\mathsf{M}_0}_{x_1}$ is a set, $\mathsf{Wo}_\lhd(g_{x_0^h}(\alpha))$ yields $\mathsf{TI}_\lhd(\mathcal{C}^{\mathsf{M}_0}_{x_1}, g_{x_0^h}(\alpha))$, and further, $\mathsf{Wo}^{\mathsf{M}_0}_\lhd(g_{x_0^h}(\alpha))$. Hence $\mathsf{TI}_\lhd(\mathsf{U}, g^{c^h}(g_{x_0^h}(\alpha)))$ follows by $(*)$. $\square$

V.2 Finitary Tait-style systems

A Tait-style system for a theory S derives finite sets Γ, Δ, Λ of $\mathcal{L}$-formulas, also referred to as *sequents* (cf. e.g. Tait [30]). We write A instead of $\{A\}$; Γ, A for $\Gamma \cup \{A\}$ and Γ, Δ to abbreviate $\Gamma \cup \Delta$. Further, for $i \in \{0,1\}$, $\mathsf{FV}_i(\Gamma) := \bigcup\{\mathsf{FV}_i(A) : A \in \Gamma\}$. In similar fashion, we lift the functions $\neg A$, $A[\mathcal{X}/\mathcal{Y}]$, and $A{\restriction}\mathcal{C}$ from formulas to sequents: for instance $\Gamma{\restriction}\mathcal{C} := \{A{\restriction}\mathcal{C} : A \in \Gamma\}$.

We map sequents to formulas as follows.

Definition V.2.1. *If $\Gamma := \{A_0, \ldots, A_{k-1}\}$, then $(\Gamma) := \bigvee_{i<k} A_{\pi(i)}$, where π is a permutation on $\{0, \ldots, k-1\}$ so that $\ulcorner A_{\pi(0)} \urcorner < \ldots < \ulcorner A_{\pi(k-1)} \urcorner$, and $\ulcorner A \urcorner$ is the Gödelnumber of A. We consider two sequents Γ and Δ as equivalent over some theory T, if the corresponding formulas (Γ) and (Δ) are equivalent over T. If $\Gamma = \emptyset$, then $(\Gamma) := \bot$.*

The notion $\mathcal{C} \models A$ is lifted to sequents as follows.

Definition V.2.2. *Let $\mathcal{C}$ be a class term of $\mathcal{L}$, Γ a sequent of $\mathcal{L}$-formulas with $\mathsf{FV}_1(\Gamma) \setminus \mathsf{FV}_1(\mathcal{C}) = \{V_1, \ldots, V_n\}$ and Var a finite set of number variables. Then,*

$$\mathcal{C} \models_{Var} \Gamma := \Delta[(\mathcal{C})_{v_1}/V_1, \ldots, (\mathcal{C})_{v_n}/V_n], \text{ where } \Delta := \Gamma{\restriction}\mathcal{C},$$

and $v_1, \ldots, v_n$ are the first variables w.r.t. some fixed enumeration that are not in Var and do not occur in $\Gamma{\restriction}\mathcal{C}$. Further, $\mathcal{C} \models \Gamma := \mathcal{C} \models_\emptyset \Gamma$.

Note that due to the way we pick the fresh variables $v_1, \ldots, v_n$, the sequent $\mathcal{C} \models \Gamma \cup \Delta$ may be different from the sequent $\mathcal{C} \models \Gamma, \mathcal{C} \models \Delta$, which would be annoying. However, if Var is the set of number variables that occur in $(\Gamma, \Delta){\restriction}\mathcal{C}$, then $\mathcal{C} \models \Gamma \cup \Delta$ is $\mathcal{C} \models_{\text{Var}} \Gamma \cup \Delta$ is $\mathcal{C} \models_{\text{Var}} \Gamma, \mathcal{C} \models_{\text{Var}} \Delta$. Thus, if a sequent Γ, Δ is given, then we read $\mathcal{C} \models \Gamma, \mathcal{C} \models \Delta$ as $\mathcal{C} \models_{\text{Var}} \Gamma, \mathcal{C} \models_{\text{Var}} \Delta$, where Var is a above.

All finitary Tait-style systems considered in this thesis share the axioms and rules of the Tait-style system for T^ϵ: for each atom A, each sequent $\Gamma \supseteq \{A, {\sim}A\}$ is an axiom with main-formulas A and ${\sim}A$. Besides, we have axioms for the primitive recursive function and relation symbols. We assume that the main-formulas of these axioms consist of literals only and are closed under substitution, i.e. if $\Gamma(\vec{u})$ is an axiom, then so is $\Gamma(\vec{s})$ for all number terms $\vec{s}$. The rules are the usual rules for conjunction, disjunction and quantification in both sorts displayed below. $A(v), B(V), A_1, A_2, C$ and the elements of Γ range over $\mathcal{L}$-formulas and s, t over $\mathcal{L}$-terms.

$$\frac{\Gamma, A(s)}{\Gamma, \exists x A(x)}, \qquad \frac{\Gamma, A(u)}{\Gamma, \forall x A(x)}, \qquad\qquad \frac{\Gamma, B(U)}{\Gamma, \exists X B(X)}, \qquad \frac{\Gamma, B(U)}{\Gamma, \forall X B(X)},$$

where $U \notin \mathsf{FV}_1(\Gamma, \forall X B(X))$ and $u \notin \mathsf{FV}_0(\Gamma, \forall x A(x))$ is required for the $\forall$-rules. Additionally, we have

$$\frac{\Gamma, A_1, A_2}{\Gamma, A_1 \vee A_2}, \quad \frac{\Gamma, A_1 \qquad \Gamma, A_2}{\Gamma, A_1 \wedge A_2}, \quad \text{and} \quad \frac{\Gamma, C \qquad \Gamma, \neg C}{\Gamma} \quad \text{(cut)}.$$

A formula displayed beside Γ in the conclusion of the above rules is called the main-formula of this rule, and the formulas C and $\neg C$ displayed in the cut-rule are referred to as cut-formulas. Restricting the cut-rule to a certain set $\mathbb{C}$ of formulas means that the cut-rule is only applicable if one of the cut-formulas is in $\mathbb{C}$.

Next, we assign a Tait-style system to a theory $\mathsf{S} := \mathsf{T}^\epsilon + \breve{\mathsf{S}}$. We could consider the extension of T^ϵ by all sequents that contain the formula $\breve{\mathsf{S}}$. However, for reasons related to cut-elimination, we do not want the main-formula of an axiom to be overly complex. Instead of $\breve{\mathsf{S}}$, we consider the $\mathcal{L}$-instances $\mathsf{inst}(\breve{\mathsf{S}})$ of the theory S.

Definition V.2.3. *The $\mathcal{L}$-instances $\mathsf{inst}(A)$ of an $\mathcal{L}$-formula A is the set of $\mathcal{L}$-formulas inductively defined as follows.*

(i) If A is not of the form $\forall X A'$, $\forall x A'$ or $A_0 \wedge A_1$, then $\mathsf{inst}(A) := \{A\}$.

(ii) $\mathsf{inst}(A \wedge B) := \mathsf{inst}(A) \cup \mathsf{inst}(B)$, $\mathsf{inst}(\forall X A(X)) := \bigcup_{i \in \mathbb{N}} \mathsf{inst}(A(U_i))$ and $\mathsf{inst}(\forall x A(x)) := \bigcup_{s \in \mathcal{L}} \mathsf{inst}(A(s))$, where $s \in \mathcal{L}$ states that s is an $\mathcal{L}$-term.

We refer to $\mathsf{inst}(\breve{\mathsf{S}})$ as the $\mathcal{L}$-instances of the theory S.

Example V.2.4. $\mathsf{p}_1(\breve{\mathsf{T}}) := \forall Z \exists X [Z \dot{\in} X \wedge \breve{\mathsf{T}} \restriction X] \wedge \mathsf{pair} \wedge \mathsf{trans}$. *The instances of $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S}$ are thus the following:*

(i) the instances of pair and trans, that is, $\exists Y[Y = (U)_s]$ and $\exists Z[Z = U{+}V]$ (where $U{+}V := \{\langle x, 0\rangle : x \in U\} \cup \{\langle y, z{+}1\rangle : \langle y, z\rangle \in V\}$), for all set variable U, V and each number term s.

(ii) $\exists X[U \dot{\in} X \wedge \breve{\mathsf{T}} \restriction X]$ for each set variable U, and $\exists X[X = \mathsf{M}_0]$, unless S is T^ϵ.

(iii) the instances of $\breve{\mathsf{S}} \restriction \mathsf{M}_0$, unless S is T^ϵ.

Observe that (i) and (ii) are Σ^1_1, and (iii) are arithmetical.

Definition V.2.5. *The Tait-style system assigned to a theory* $\mathsf{S} := \mathsf{T}^\epsilon + \breve{\mathsf{S}}$ *extends then* T^ϵ *by axioms* Γ, A *for each* $A \in \mathsf{inst}(\breve{\mathsf{S}})$, *where* A *is the main-formula of this axiom. And* $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma$ *states that there is a derivation of the sequent* Γ *of depth* n, *where the cut-rule is only applied if one of the distinguished formulas of the cut-rule is an element of* $\mathsf{inst}(\breve{\mathsf{S}})$.

The next result states some standard inversion properties of Tait-style systems. Observe that formulas of the form $A \wedge B$, $\forall X A(X)$ and $\forall x B(x)$ are not $\mathcal{L}$-instances of $\breve{\mathsf{S}}$ and are thus not main-formulas of axioms of S.

Lemma V.2.6. *Assume that the displayed formula* $A \vee B$ *below is not an* $\mathcal{L}$*-instances of* S. *Then, we have the following.*

(i) If $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, \forall X A(X)$ *and* $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, \forall x B(x)$, *then also* $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, A(U_i)$ *and* $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, B(s)$, *for each* $i \in \mathbb{N}$ *and each* $s \in \mathcal{L}$.

(ii) If $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, A \vee B$, *then also* $\mathsf{S} \,|\!\frac{n}{*}\, \Gamma, A, B$.

Having a notion of derivation at hand allows us to also give a proof-theoretic proof of Lemma I.1.9.

Lemma V.2.7. *For each class term* $\mathcal{C}$, $\mathsf{T}^\epsilon|\mathsf{S} \vdash \Gamma \Rightarrow \mathsf{T}^\epsilon|\mathsf{S} \vdash \mathcal{C} \models \Gamma$.

Proof By induction on the depth of the derivation. The only axiom of $\mathsf{T}^\epsilon|\mathsf{S}$, where the main-formulas contain set variables is $t \in U, \neg t \notin U$. As $C(t), \neg C(t)$ is derivable in T^ϵ for each formula C, also $t \in (\mathcal{C})_u, t \notin (\mathcal{C})_u$ is provable. And if e.g. $\Gamma, \forall X A(X)$ with $\mathsf{FV}_1(\Gamma, \forall X A(X)) = \{V_1, \ldots, V_k\}$ is obtained form $\Gamma, A(U)$ with $U \notin \mathsf{FV}_1(\Gamma, \forall X A(X))$ by a $\forall X$-rule (we may also assume that $U \notin \mathsf{FV}_1(\mathcal{C})$), then by I.H. $\Gamma \restriction \mathcal{C}[\mathcal{C}_{\vec{v}}/\vec{V}], A(U) \restriction \mathcal{C}[\mathcal{C}_u/U, \mathcal{C}_{\vec{v}}/\vec{V}]\}$. As $\vec{v}, u$ are fresh and pairwise distinct variables, a $\forall x$-rule yields $\Gamma \restriction \mathcal{C}[\vec{V}/\mathcal{C}_{\vec{v}}], \forall x(A(U) \restriction \mathcal{C}[(\mathcal{C})_x/U, \mathcal{C}_{\vec{v}}/\vec{V}])$. As $(\forall X A(X)) \restriction \mathcal{C}$ is $\forall x(A(U) \restriction \mathcal{C}[(\mathcal{C})_x/U])$, the claim follows. The other cases are similar or simpler. □

The following technical lemma will be used in the proof of Lemma VI.2.3, which is a key step in the reduction of $\mathsf{p}_1(\mathsf{T}) \,|\!\frac{n}{*}\, \Gamma$ to $\mathsf{T}^n \vdash \Gamma$ for an arithmetical Γ. Since $\mathsf{T}|\mathsf{T}^{n+2}$ implies that $\breve{\mathsf{T}} \restriction \mathsf{U}_{n+2}$, but also that $\breve{\mathsf{T}} \restriction \mathsf{U}_{n+1}$, we can, under suitable assumptions, substitute U_{n+2} for U_{n+1}. Further, we have added a consequence of the above result.

Lemma V.2.8.

(i) $\mathsf{S} \vdash \Gamma \Longrightarrow \mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \models \Gamma$.

(ii) if $\Gamma \subseteq \mathcal{L}(\mathsf{T}^{n+2}|\mathsf{S})$, *then*

$$\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \Gamma \Longrightarrow \mathsf{T}^\epsilon|\mathsf{T}^{n+2}|\mathsf{S} \vdash \Gamma[\mathsf{M}_{n+2}/\mathsf{M}_{n+1}, \ldots, \mathsf{M}_2/\mathsf{M}_1].$$

(iii) if $\Gamma \subseteq \mathcal{L}(\mathsf{T}^2|\mathsf{S})$*, then* $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \Gamma \Longrightarrow \mathsf{T}^\epsilon|\mathsf{T}^{n+2}|\mathsf{S} \vdash \Gamma[\mathsf{M}_2/\mathsf{M}_1]$.

Proof (i) If S is T^ϵ, this is by Lemma V.2.7. Else, $\mathsf{S} \vdash \Gamma$ implies $\mathsf{T}^\epsilon \vdash \neg\breve{\mathsf{S}}, \Gamma$, and thus $\mathsf{T}^\epsilon \vdash \neg\breve{\mathsf{S}}{\restriction}\mathsf{M}_0, \mathsf{M}_0 \models \Gamma$ by the Lemma V.2.7. Hence $\mathsf{T}^\epsilon + \breve{\mathsf{S}}{\restriction}\mathsf{M}_0 \vdash \mathsf{M}_0 \models \Gamma$, that is $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \models \Gamma$. (ii) We show that $\mathsf{T}^\epsilon|\mathsf{T}^{n+2}|\mathsf{S} \vdash \Gamma[\mathsf{M}_{n+2}/\mathsf{M}_{n+1}, \ldots, \mathsf{M}_2/\mathsf{M}_1]$ for each axiom Γ of $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S}$. Then the claim follows easily by induction on the depth of the derivation. The non-trivial case is if the main-formula of Γ is $A_n := \breve{\mathsf{T}}^{n+1}|\mathsf{S}{\restriction}\mathsf{M}_{n+1}$. We show by induction on n, that $\mathsf{T}^\epsilon|\mathsf{T}^{n+2}|\mathsf{S} \vdash A'_n := A_n[\mathsf{M}_{n+2}/\mathsf{M}_{n+1}, \ldots, \mathsf{M}_2/\mathsf{M}_1]$, that is, $\breve{\mathsf{T}}^{n+2}|\mathsf{S}{\restriction}\mathsf{M}_{n+2}$ implies A'_n.
$n = 0$. $\breve{\mathsf{T}}^2|\mathsf{S}{\restriction}\mathsf{M}_2$ implies $\breve{\mathsf{T}}{\restriction}\mathsf{M}_2$ and $\mathsf{M}_0 \dot{\in} \mathsf{M}_1 \dot{\in} \mathsf{M}_2$ and $\breve{\mathsf{T}}|\mathsf{S}{\restriction}\mathsf{M}_1$ and $\breve{\mathsf{S}}{\restriction}\mathsf{M}_0$. Since M_2 satisfies (ACA), M_2 is transitive, so $\mathsf{M}_0 \dot{\in} \mathsf{M}_2$, and $\breve{\mathsf{T}}|\mathsf{S}{\restriction}\mathsf{M}_2$, that is, $A_0[\mathsf{M}_2/\mathsf{M}_1]$ i.e. A'_0. Next, assume that $n > 0$ and $\breve{\mathsf{T}}^{n+2}|\mathsf{S}{\restriction}\mathsf{M}_{n+2}$ implies A'_n. Further observe that $A_{n+1} = \breve{\mathsf{T}}{\restriction}\mathsf{M}_{n+2} \wedge \breve{\mathsf{T}}^{n+1}|\mathsf{S}{\restriction}\mathsf{M}_{n+1}$, so $A'_{n+1} = \breve{\mathsf{T}}{\restriction}\mathsf{M}_{n+3} \wedge \breve{\mathsf{T}}^{n+1}|\mathsf{S}{\restriction}\mathsf{M}_{n+1}[\mathsf{M}_{n+2}/\mathsf{M}_{n+1}, \ldots, \mathsf{M}_2/\mathsf{M}_1]$, which is $\breve{\mathsf{T}}{\restriction}\mathsf{M}_{n+3} \wedge A'_n$. Now the induction step follows as $\breve{\mathsf{T}}^{n+3}|\mathsf{S}{\restriction}\mathsf{M}_{n+3}$ implies $\breve{\mathsf{T}}{\restriction}\mathsf{M}_{n+3}$ and $\breve{\mathsf{T}}^{n+2}|\mathsf{S}{\restriction}\mathsf{M}_{n+2}$, which by I.H. yields A'_n.
(iii) is immediate by (ii) as $\mathsf{M}_{n+2}, \ldots, \mathsf{M}_2$ do not occur in $\Gamma \subseteq \mathcal{L}(\mathsf{T}^2|\mathsf{S})$. □

V.3 Infinitary Tait-style systems

An infinitary Tait-style system derives finite sets of $\mathcal{L}^*$-formulas ($\mathcal{L}$-formulas without free number variables) usually denoted by Γ or Δ. We write $s \in \mathcal{L}^*$ to indicate that s is a closed number term of $\mathcal{L}^*$ and $A \in \mathcal{L}^*$ to indicate that A is an $\mathcal{L}^*$-formula. Accordingly, $\mathcal{L}^*_{\Pi^1_0}(\mathsf{S})$ are the formulas in $\mathcal{L}_{\Pi^1_0}(\mathsf{S})$ without free number variables. Further, an $\mathcal{L}$-formula A is identified with the $\mathcal{L}^*$-formula A^* obtained from A by replacing each free number variable u_n by the numeral $\overline{n}$.

An infinitary Tait-style system imports the natural numbers $\mathbb{N}$ from the meta-theory into the infinitary system via the ω-rule, which asserts that $\forall x A(x)$ holds if $A(\overline{n})$ holds for all $n \in \mathbb{N}$. As a consequence (cf. Pohlers [11]), the infinitary system $\overset{*}{\mathsf{T}}{}^\epsilon$ is complete w.r.t. Π^1_1-sentences of $\mathcal{L}^*(\mathsf{T})$: if for each interpretation $\mathcal{U}^{\mathbb{N}} \in \mathcal{P}(\mathbb{N})$ of the relation symbol U, the Π^1_1-sentence A is valid in the (standard) model of arithmetic $(\mathcal{P}(\mathbb{N}), \mathbb{N}, \mathcal{U}^{\mathbb{N}})$, then $\overset{*}{\mathsf{T}}{}^\epsilon \vdash A$.

In the standard model $(\mathbb{N}, \ldots)$, each $s \in \mathcal{L}^*$ evaluates to a natural number $s^{\mathbb{N}}$. Two $\mathcal{L}^*$-formulas are *numerically equivalent* if they are syntactically equivalent modulo number terms which have the same value. Literals of $\mathcal{L}^*$ that do not contain the relation symbols $(\mathsf{U}_i : i \in \mathbb{N})$ and $\in$ evaluate to true or false in the standard model. Those that evaluate to true are referred to as the *true* literals of $\mathcal{L}^*$, and those that evaluate to false are referred to as the *false* literals.

The axioms of $\overset{*}{\mathsf{T}}{}^\epsilon$ are all the sequents of $\mathcal{L}^*$-formulas of the form Γ, A and $\Gamma, B, {\sim}C$, where A is a true literal and B and C are numerically equivalent literals. The rules of $\overset{*}{\mathsf{T}}^\epsilon$ are the rules of T^ϵ restricted to $\mathcal{L}^*$-formulas, but with the $\forall x$-rule replaced by the following rule, called ω-rule,

$$(\omega\text{-rule}) \qquad \frac{\Gamma, A(\overline{n}) \text{ for all } n \in \mathbb{N}}{\Gamma, \forall x A(x)}.$$

An infinitary Tait-style system corresponding to a finitary theory S then extends T^ϵ by axioms Γ, A, where A is now an $\mathcal{L}^*$-instances $\mathsf{inst}^*(\breve{\mathsf{S}})$ of S. The $\mathcal{L}^*$-instances of a formula A are defined similarly as the $\mathcal{L}$-instances, but we close $\mathsf{inst}^*(A)$ under numerical equivalence, and further, the definition is so that if A is a true literal, then $\mathsf{inst}^*(A \to B) := \mathsf{inst}^*(B)$. With these modifications, the $\mathcal{L}^*$-instances of the infinitary systems $\overset{*}{\mathsf{T}}_x$ ($x \in Q$) are simple enouth to allow for a lean cut-elimination.

Definition V.3.1. *The $\mathcal{L}^*$-instances $\mathsf{inst}^*(A)$ of an $\mathcal{L}^*$-formula A are inductively defined as follows:*

(i) *If A is true literal or B is a true literal, then $\mathsf{inst}^*(A \vee B) := \emptyset$. If A is a false literal and B is not a true literal, then $\mathsf{inst}^*(A \vee B) := \mathsf{inst}^*(B \vee A) := \mathsf{inst}^*(B)$.*

(ii) $\mathsf{inst}^*(A \wedge B) := \mathsf{inst}^*(A) \cup \mathsf{inst}^*(B)$, $\mathsf{inst}^*(\forall X A(X)) := \bigcup_{i \in \mathbb{N}} \mathsf{inst}^*(A(U_i))$, *and* $\mathsf{inst}^*(\forall x A(x)) := \bigcup_{s \in \mathcal{L}^*} \mathsf{inst}^*(A(s))$.

(iii) $\mathsf{inst}^*(A) := \{B : A \text{ and } B \text{ are numerically equivalent}\}$, *if neither (i) nor (ii) applies.*

We refer to $\mathsf{inst}^*(\breve{\mathsf{S}})$ as the $\mathcal{L}^*$-instances of theory S.

Definition V.3.2. *If S is a theory, different form $\overset{*}{\mathsf{T}}^\epsilon$, then the corresponding infinitary Tait-style system $\overset{*}{\mathsf{S}}$ extends $\overset{*}{\mathsf{T}}^\epsilon$ by axioms Γ, A, where Γ is a finite set of $\mathcal{L}^*$-formulas and $A \in \mathsf{inst}^*(\breve{\mathsf{S}})$. Further, A is the main-formula of this axiom.*

Next, we define when $\overset{*}{\mathsf{S}} \,\frac{\alpha}{\mathbb{C}}\, \Gamma$, that is, when $\overset{*}{\mathsf{S}}$ proves a sequent Γ with depth α and the cut-rule restricted to the set

$$\mathsf{cut}(\mathbb{C}, \mathsf{S}) := \{A(U) \in \mathcal{L}^*(\mathsf{S}) : A(V) \in \mathbb{C} \cup \neg\mathbb{C}\} \cup \mathsf{inst}^*(\breve{\mathsf{S}}) \cup \neg\mathsf{inst}^*(\breve{\mathsf{S}}),$$

where $\mathsf{inst}^*(\breve{\mathsf{T}}^\epsilon) := \emptyset$. Observe that to obtain $\mathsf{cut}(\mathbb{C}, \mathsf{S})$ from $\mathbb{C}$, we first close under substitution of set variables and restrict to formulas in $\mathcal{L}^*(\mathsf{S})$, and then add the $\mathcal{L}^*$-instances of S and their negations since we cannot avoid these cuts.

Definition V.3.3. *For all ordinals α and each set $\mathbb{C}$ of $\mathcal{L}^*$-formulas, we define $\overset{*}{\mathsf{S}} \,\frac{\alpha}{\mathbb{C}}\, \Gamma$ by recursion on α as follows.*

(i) *If Γ is an axiom of $\overset{*}{\mathsf{S}}$, then $\overset{*}{\mathsf{S}} \,\frac{\alpha}{\mathbb{C}}\, \Gamma$ for all ordinals α.*

(ii) *If for all $i \in I$, $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha_i}{\mathbb{C}}\, \Gamma, A_i$ and $\alpha_i < \alpha$ for all premises Γ, A_i of a rule that is not a cut, then $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A$ for the conclusion of this rule.*

(iii) *If $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha_1}{\mathbb{C}}\, \Gamma, A$ and $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha_2}{\mathbb{C}}\, \Gamma, \neg A$ with $\alpha_1, \alpha_2 < \alpha$, and $\{A, \neg A\} \subseteq \mathsf{cut}(\mathbb{C}, \mathsf{S})$, then $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma$.*

$\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\quad}\, \Gamma$ states $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathcal{L}^*}\, \Gamma$, where $\mathcal{L}^*$ denotes the set of all $\mathcal{L}^*$-formulas, and $\overset{*}{\mathsf{S}} \,|\!\frac{<\beta}{\mathbb{C}}\, \Gamma$ states that $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma$ for some $\alpha < \beta$. Further, we write $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{*}\, \Gamma$ for $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\emptyset}\, \Gamma$, and $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{-}\, \Gamma$ states that Γ is obtained without using the cut-rule at all.

We point out that $\overset{*}{\mathsf{T}} \,|\!\frac{\quad}{*}\, \Gamma$ is $\overset{*}{\mathsf{T}} \,|\!\frac{\quad}{-}\, \Gamma$, so the derivation is cut-free, while if S is different from T^ϵ, then the derivation $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \,|\!\frac{\quad}{*}\, \Gamma$ still contains cuts with $\mathcal{L}^*$-instances of $\mathsf{T}^\epsilon|\mathsf{S}$, that is, formulas in $\mathsf{inst}^*(\breve{\mathsf{S}}{\restriction}\mathsf{M}_0)$.

Although the following properties of Tait-style systems are essentially trivial, we consider them important enough to summarize them in a lemma. Note that the axioms of $\overset{*}{\mathsf{S}}$ are by design closed under numerical equivalence.

Lemma V.3.4. *Let $s^{\mathbb{N}} = t^{\mathbb{N}}$, and B a false literal. Then,*

(i) *$\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A(s)$ iff $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A(t)$,*

(ii) *$\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, B$ iff $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma$.*

Next, we state inversion properties of Tait-style systems. Observe that formulas of the form $A \wedge B$, $\forall X A(X)$ and $\forall x B(x)$ are not $\mathcal{L}^*$-instances of $\breve{\mathsf{S}}$ and are thus not main-formulas of axioms of $\overset{*}{\mathsf{S}}$.

Lemma V.3.5.

(i) *If $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, \forall X A(X)$ and $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, \forall x B(x)$, then also $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A(U_i)$ and $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, B(s)$, for each $i \in \mathbb{N}$ and each $s \in \mathcal{L}^*$.*

(ii) *If $A \vee B \notin \mathsf{inst}^*(\breve{\mathsf{S}})$ and $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A \vee B$, then $\overset{*}{\mathsf{S}} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, A, B$.*

Also the following simple observation is occasionally useful. It is often used tacitly if we prove results by induction on the depth of the derivation.

Lemma V.3.6. *If $\overset{*}{\mathsf{S}} \,|\!\frac{\gamma}{\mathbb{C}}\, \Gamma$ is not obtained by an ω-rule, then $\overset{*}{\mathsf{S}} \,|\!\frac{<\gamma}{\mathbb{C}}\, \Gamma$.*

Proof If $\overset{*}{\mathsf{S}} \,|\!\frac{\gamma}{\mathbb{C}}\, \Gamma$ is not obtained by an ω-rule, then the last rule applied has at most two premises which are derivable with depth $\alpha < \gamma$. As then also $\alpha+1 < \gamma$, the claim follows. $\square$

Next, we show kind of an infinitary deduction theorem, see Lemma V.3.9. Here, and often in the sequel, it suffice to bound the depth of a derivation by a limit ordinal. Thus, the following definition.

Definition V.3.7. *For each α, $\alpha^+ := \alpha{+}\omega$.*

If $A \in \mathsf{inst}^*(B)$, then $\overset{*}{\mathsf{T}}{}^\epsilon \vdash B \to A$ and $\overset{*}{\mathsf{T}}{}^\epsilon \vdash \neg A \to \neg B$. Essentially, this is behind the next auxiliary Lemma.

Lemma V.3.8. *If $A \in \mathsf{inst}^*(B)$ and $\overset{*}{\mathsf{S}} \vdash^{\alpha}_{\mathbb{C}} \Gamma, \neg A$, then $\overset{*}{\mathsf{S}} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg B$.*

Proof By induction on the build-up of B. Let $A \in \mathsf{inst}^*(B)$. If B is a true literal, then $\neg A$ is a false literal, and the claim is by Lemma V.3.4; if B is a false literal, then $\neg A$ is a true literal, and the claim holds trivially. And if e.g. B is of the form $B_1 \wedge B_2$, then e.g. $A \in \mathsf{inst}^*(B_1)$. By I.H., $\overset{*}{\mathsf{S}} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg B_1$, so also $\overset{*}{\mathsf{S}} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg B_1 \vee \neg B_2$. The other cases are shown similarly. □

Lemma V.3.9. *If $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma$, then $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \neg\breve{\mathsf{T}}, \Gamma$.*

Proof Immediate by induction on α. If e.g. $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha+1}_{\mathbb{C}} \Gamma$ is obtained from $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, \neg A$ for $A \in \mathsf{inst}^*(\breve{\mathsf{T}}|\mathsf{S})$ by a cut, then by I.H. $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \neg\breve{\mathsf{T}}, \Gamma, \neg A$, thus by Lemma V.3.8, $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \neg\breve{\mathsf{T}}, \Gamma$. □

Further, we give an infinitary variant of Lemma I.1.9 for the case where $\mathcal{C}$ is a set variable or a relation symbol. When we apply it, the set $\mathbb{D}$ in the formulation of the lemma which specifies the range of the cut-rule is $\emptyset$ or $\mathcal{L}^*_{\mathsf{e}\Sigma^1_n}$, so the assumed closure properties are met. For its proof, we use the following auxiliary result.

Lemma V.3.10. *Suppose that $\mathcal{C}$ is a set variable or a relation symbol of $\mathcal{L}^*(\mathsf{T}|\mathsf{S})$, and that with $A \in \mathbb{D}$ also $A[(\mathcal{C})_s/U] \in \mathbb{D}$. Then, $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{\alpha}_{\mathbb{D}} \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{\alpha}_{\mathbb{D}} \Gamma[(\mathcal{C})_s/U]$.*

Proof Straightforward by induction on α. Since the main-formula of an $\mathcal{L}^*$-instance of $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S}$ does not contain free set variables, the claim clearly holds for axioms. □

Lemma V.3.11. *Suppose that $\mathcal{C}$ is a set variable or a relation symbol of $\mathcal{L}^*(\mathsf{T}|\mathsf{S})$, and that for each $s \in \mathcal{L}^*$, $\mathbb{D}$ contains $(\mathcal{C})_s \neq (\mathcal{C})_s$ and with $A \in \mathbb{D}$ also $A[(\mathcal{C})_s/U] \in \mathbb{D}$. If $\mathcal{C} \notin \mathsf{FV}_1(\Gamma) = \{Z_1, \ldots, Z_l\}$ and $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{\alpha}_{\mathbb{D}} \Gamma$, then $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{D}\restriction\mathcal{C}} \vec{Z} \mathrel{\dot\notin} \mathcal{C}, \Gamma\restriction\mathcal{C}$.*

Proof By induction on α. We just show the case where $\Gamma = \Delta, \forall X A(X)$ and $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{\alpha+1}_{\mathbb{D}} \Gamma$ is obtained from $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{\alpha}_{\mathbb{D}} \Delta, A(U)$, where $U \notin \mathsf{FV}_1(\Gamma)$ is different form $\mathcal{C}$. By I.H., $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{D}\restriction\mathcal{C}} U \mathrel{\dot\notin} \mathcal{C}, \vec{Z} \mathrel{\dot\notin} \mathcal{C}, \Delta\restriction\mathcal{C}, (A\restriction\mathcal{C})(U)$. As $U \mathrel{\dot\notin} \mathcal{C}$ is $\forall x[U \neq (\mathcal{C})_x]$, inversion yields for each $s \in \mathcal{L}^*$, $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{D}\restriction\mathcal{C}} U \neq (\mathcal{C})_s, \vec{Z} \mathrel{\dot\notin} \mathcal{C}, \Delta\restriction\mathcal{C}, (A\restriction\mathcal{C})(U)$. Using

Lemma V.3.10 to substitute $(\mathcal{C})_s$ for U, cutting with $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \mid\!\frac{<\omega}{\mathbb{D}}\ (\mathcal{C})_s = (\mathcal{C})_s$, and applying an ω-rule yields $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \mid\!\frac{<\alpha^+}{\mathbb{D}\restriction\mathcal{C}}\ \vec{Z} \mathrel{\dot{\notin}} \mathcal{C}, \Delta\restriction\mathcal{C}, \forall x((A\restriction\mathcal{C})((\mathcal{C})_x))$, that is, $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \mid\!\frac{<\alpha^+}{\mathbb{D}\restriction\mathcal{C}}\ \vec{Z} \mathrel{\dot{\notin}} \mathcal{C}, \Gamma\restriction\mathcal{C}$. □

Also for later use, we state this substitution property.

Lemma V.3.12. *If* $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma[X/U]$, *then also* $\overset{*}{\mathsf{S}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ X \neq \mathcal{C}, \Gamma[\mathcal{C}/U]$.

Proof By induction on α. Note that $\mathsf{T}^\epsilon \mid\!\frac{}{-}\ (\mathcal{C})_t \neq X, s \in (\mathcal{C})_t, s \notin X$, this handles the case of an axiom of the form $\Gamma, s \in X, s \notin X$. If $A(X) \in \mathsf{inst}^*(\breve{\mathsf{S}})$, $\mathsf{T}^\epsilon \mid\!\frac{}{-}\ X \neq \mathcal{C}, \neg A(X), A(\mathcal{C})$, so $\overset{*}{\mathsf{S}}^\epsilon \mid\!\frac{<\omega}{-}\ X \neq \mathcal{C}, \neg A(X), A(\mathcal{C})$, and $\overset{*}{\mathsf{S}} \mid\!\frac{<\omega}{*}\ X \neq \mathcal{C}, A(\mathcal{C})$ follows by a cut. The induction step is straightforward (if $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha+1}{\mathbb{D}}\ \Gamma[X/U]$ is obtained from $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma[X/U], A[X/U]$ by a cut, then note that for $B := A[X/U]$, also $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma[X/U], B[X/U]$, and by I.H. $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma[\mathcal{C}/U], B$ since U does not occur in B; in the same way we obtain $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma[X/U], \neg B$). □

Before we review the basic facts about partial cut-elimination, we fix a notion of subformulas and rank of a formula.

Definition V.3.13. *The set* $\mathsf{sufo}(A)$ *of subformulas of an* $\mathcal{L}^*$*-formula* A *is defined as follows:*

(i) $\mathsf{sufo}(A) := \{A\}$, *if* A *is a literal.*

(ii) $\mathsf{sufo}(A\, j\, B) := \{AjB\} \cup \mathsf{sufo}(A) \cup \mathsf{sufo}(B)$, *where* $j \in \{\wedge, \vee\}$,

(iii) $\mathsf{sufo}(\mathcal{Q}xA(x)) := \{\mathcal{Q}xA(x)\} \cup \bigcup\{\mathsf{sufo}(A(s)) :\ s \in \mathcal{L}^*\ \}$,

(iv) $\mathsf{sufo}(\mathcal{Q}XA(X)) := \{\mathcal{Q}XA(X)\} \cup \bigcup\{\mathsf{sufo}(A(U_i)) : i \in \mathbb{N}\}$.

Further, $\mathsf{sufo}^-(A) := \mathsf{sufo}(A) - \{A\}$ *are the proper subformulas of* A. *Moreover, for a set* $\mathbb{C} \subseteq \mathcal{L}^*$, $\mathsf{sufo}(\mathbb{C}) := \bigcup_{A\in\mathbb{C}} \mathsf{sufo}(A)$ *and* $\mathsf{sufo}^-(\mathbb{C}) := \bigcup_{A\in\mathbb{C}} \mathsf{sufo}^-(A)$.

Definition V.3.14. *To each* $\mathcal{L}$*-formula, we assign a rank as follows. The rank* $\mathsf{rk}(L)$ *of a literal is* 1, $\mathsf{rk}(A\, j\, B) := \max(\mathsf{rk}(A), \mathsf{rk}(B))+1$, *and* $\mathsf{rk}(\mathcal{Q}xA) := \mathsf{rk}(\mathcal{Q}XA) := \mathsf{rk}(A)+1$. *Further,* $\mathsf{rk}(\mathbb{C}) < n$ *is short for* $(\forall A \in \mathbb{C})(\mathsf{rk}(A) < n)$.

Partial-cut elimination now reads as follows, where $\alpha\#\beta$ denotes the natural sum of two ordinals.

Lemma V.3.15. *Let* $\Gamma, \Delta, A \subseteq_{\mathsf{fin}} \mathcal{L}^*$, $\mathbb{C} \subseteq \mathcal{L}^*$ *and assume that* $\mathsf{sufo}^-(A) \subseteq \mathsf{cut}(\mathbb{C}, \mathsf{S})$ *and* $\mathsf{sufo}^-(\mathbb{C}) \subseteq \mathsf{sufo}(\mathbb{D})$.

(i) If $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha}{\mathbb{C}}\ \Gamma, A$ *and* $\overset{*}{\mathsf{S}} \mid\!\frac{\beta}{\mathbb{C}}\ \Delta, \neg A$, *then* $\overset{*}{\mathsf{S}} \mid\!\frac{\alpha\#\beta}{\mathbb{C}}\ \Gamma$.

(ii) $\overset{*}{\mathsf{S}} \vdash^{\alpha}_{\mathbb{C}} \Gamma \Longrightarrow \overset{*}{\mathsf{S}} \vdash^{2^\alpha}_{\mathbb{D}} \Gamma$ *and* $\overset{*}{\mathsf{S}} \vdash^{\alpha} \Gamma \Longrightarrow \overset{*}{\mathsf{S}} \vdash^{E(\alpha)}_{*} \Gamma$, *where* $E = \{\alpha : \alpha = \omega^\alpha\}$.

Proof We only provide some hints. (i) We consider the case where $A := \forall x B(x)$ with $B(\overline{m}) \in \mathsf{cut}(\mathbb{C}, \mathsf{S})$ for all $m \in \mathbb{N}$. Assuming that A and $\neg A$ are the main-formulas of the last inference, we have for some $\alpha' < \alpha$, $\beta' < \beta$, some $s \in \mathcal{L}^*$ with $s^{\mathbb{N}} = m$, that $\overset{*}{\mathsf{S}} \vdash^{\alpha'}_{\mathbb{C}} \Gamma, A, B(\overline{m})$ and $\overset{*}{\mathsf{S}} \vdash^{\beta'}_{\mathbb{C}} \Delta, \neg A, \neg B(s)$. Hence the I.H. and Lemma V.3.5 (iii) yield $\overset{*}{\mathsf{S}} \vdash^{\alpha \# \beta'}_{\mathbb{C}} \Gamma, \Delta, \neg B(s)$ and $\overset{*}{\mathsf{S}} \vdash^{\alpha' \# \beta}_{\mathbb{C}} \Gamma, \Delta, B(s)$, and the claim follows by a cut. (ii) Induction on α using (i) yields the first claim, since with $A \in \mathsf{cut}(\mathbb{C}, \mathsf{S})$, either also $A \in \mathsf{cut}(\mathbb{D}, \mathsf{S})$ or $\mathsf{sufo}^{-}(A) \subseteq \mathsf{cut}(\mathbb{D}, \mathsf{S})$. Induction on α using the first claim yields the second. □

Also a simple but relevant property is the following, for whose formulation we use the notion of a substitution instance of Γ.

Definition V.3.16. *If Γ is a finite set of $\mathcal{L}$-formulas with $\mathsf{FV}(\Gamma) = \{\vec{V}, \vec{v}\}$, then for all set variables $\vec{U}$ and $\vec{s} \in \mathcal{L}^*$, $\Gamma[\vec{U}/\vec{V}, \vec{s}/\vec{v}]$ is called a substitution instance of Γ.*

Lemma V.3.17. *If $\mathsf{S} \vdash \Gamma$ with $\mathsf{FV}(\Gamma) = \{\vec{V}, \vec{v}\}$, then for all set variables $\vec{U}$ and $\vec{s} \in \mathcal{L}^*$,*

(i) $\mathsf{S} \vdash^{<\omega}_{*} \Gamma$,

(ii) $\overset{*}{\mathsf{S}} \vdash^{<\omega}_{*} \Gamma[\vec{U}/\vec{V}, \vec{s}/\vec{v}]$.

Proof (i) By partial cut-elimination and completely standard. (ii) From (i) by induction on n. For $n = 0$ the claim is readily checked. If $\mathsf{S} \vdash^{n+1}_{*} \Gamma$ is obtained by a cut, a $\mathcal{Q}X$-,$\wedge$-, $\vee$- or $\exists x$-rule, the claim follows immediately by the I.H. So assume that $\mathsf{S} \vdash^{n+1}_{*} \Gamma, \forall x A(x)$ is obtained from $\mathsf{S} \vdash^{n}_{*} \Gamma, A(u)$ with $u \notin \mathsf{FV}_0(\Gamma, \forall x A(x))$. By I.H., $\overset{*}{\mathsf{S}} \vdash^{<\omega}_{*} \Gamma', A'(\overline{n})$ for each n and each substitution instance $\Gamma', A'(\overline{n})$ of $\Gamma, A(\overline{n})$. Now $\Gamma', \forall x A'(x)$ follows by the ω-rule. □

For later reference, we collect some further auxiliary results. The next lemma helps us to deal with instances of induction and transfinite induction.

Lemma V.3.18. *Let $\mathcal{C}$ be a class term and n_0 so that $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{n_0}_{-} s \notin \mathcal{C}, s \in \mathcal{C}$ for each $s \in \mathcal{L}^*$. Then we have the following.*

(i) $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{n_0 + 2s^{\mathbb{N}}}_{-} 0 \notin \mathcal{C}, \neg B, s \in \mathcal{C}$, *for* $B := \forall x[x \in \mathcal{C} \to x{+}1 \in \mathcal{C}]$.

(ii) $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{n_0 + 4(\alpha+1)}_{-} \neg \mathsf{Prog}_{\lhd}(\mathcal{C}), \alpha \in \mathcal{C}$.

Proof (i) By induction on $s^{\mathbb{N}}$. If $s^{\mathbb{N}} = 0$, then the claim is by choice of n_0. For the induction step, assume $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{n_0 + 2s^{\mathbb{N}}}_{-} 0 \notin \mathcal{C}, \neg B, s \in \mathcal{C}$. As $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{n_0}_{-} s{+}1 \notin \mathcal{C}, s{+}1 \in \mathcal{C}$, we obtain $\overset{*}{\mathsf{T}}^{\epsilon} \overset{n_0+2s^{\mathbb{N}}+1}{\underset{-}{\vdash}} 0 \notin \mathcal{C}, \neg B, s \in \mathcal{C} \wedge s{+}1 \notin \mathcal{C}, s{+}1 \in \mathcal{C}$. An application of the $\exists$-rule yields the induction step. (ii) By induction on α. We just show the induction step. By I.H. we have for each $\beta \lhd \alpha$, $\overset{*}{\mathsf{T}}^{\epsilon} \overset{n_0+4(\beta+1)}{\underset{-}{\vdash}} \neg\mathsf{Prog}_{\lhd}(\mathcal{C}), \neg(\beta \lhd \alpha), \beta \in \mathcal{C}$. Hence,

(i) $\overset{*}{\mathsf{T}}^{\epsilon} \overset{n_0+4\alpha+2}{\underset{-}{\vdash}} \neg\mathsf{Prog}_{\lhd}(\mathcal{C}), (\forall\beta \lhd \alpha)(\beta \in \mathcal{C})$, and

(ii) $\overset{*}{\mathsf{T}}^{\epsilon} \overset{n_0+4\alpha+3}{\underset{-}{\vdash}} \neg\mathsf{Prog}_{\lhd}(\mathcal{C}), (\forall\beta \lhd \alpha)(\beta \in \mathcal{C}) \wedge \alpha \notin \mathcal{C}, \alpha \in \mathcal{C}$.

An application of the $\exists$-rule yields the induction step. $\square$

V.4 Cut-formula replacement

The result of this section is used to cheaply eliminate a cut of $\overset{*}{\mathsf{S}} \overset{n}{\underset{\mathbb{C}}{\vdash}} \Gamma, A$ with $\overset{*}{\mathsf{S}} \overset{\alpha}{\underset{\mathbb{C}}{\vdash}} \Gamma, \neg A$, if $A \in \mathsf{e}\Sigma(\mathbb{C})$. The strategy is to replace the cut-formula A by an equivalent formula $A^{\circ} \in \Sigma(\mathbb{C})$, without a significant increase of the depth of the derivation, say, $\overset{*}{\mathsf{S}} \overset{<\omega}{\underset{\mathbb{C}}{\vdash}} \Gamma, A^{\circ}$ and $\overset{*}{\mathsf{S}} \overset{<\alpha^+}{\underset{\mathbb{C}}{\vdash}} \Gamma, \neg A^{\circ}$. Then $\overset{*}{\mathsf{S}} \overset{<\alpha^+}{\underset{\mathbb{C}}{\vdash}} \Gamma$ is by Lemma V.3.15. This result is then used to provide criteria when two equivalent theories S and S' satisfy $\overset{*}{\mathsf{S}} \overset{<\alpha^+}{\underset{\mathbb{C}}{\vdash}} \Gamma$ iff $\overset{*}{\mathsf{S}}' \overset{<\alpha^+}{\underset{\mathbb{C}}{\vdash}} \Gamma$.

In the proof of the above mentioned result, we use an instance of arithmetical comprehension to code two sets U and V into one, namely $\exists X[U = (X)_0 \wedge V = (X)_1]$. Then, we need to eliminate a cut with this instance. To avoid a relevant increase of the depth of the derivation, we assume that $\mathsf{sufo}^{-}(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathbb{C}$; an assumption that is met whenever we employ Lemma V.4.2 or one of its consequences obtained in this section.

First, we assign to each $A \in \mathsf{e}\Sigma(\mathbb{C})$ a formula $A^{\circ} \in \Sigma(\mathbb{C})$ with the same free variables, so that $\mathsf{ACA}_0 \vdash A \leftrightarrow A^{\circ}$. Arithmetical comprehension is required to have $\forall X, Y A(X, Y)$ iff $\forall X A((X)_0, (X)_1)$ (or equivalently, that $\exists X, Y A(X, Y)$ iff $\exists X A((X)_0, (X)_1)$).

Definition V.4.1. *To each formula $A \in \mathsf{e}\Sigma(\mathbb{C})$, we assign a formula $A^{\circ} \in \Sigma(\mathbb{C})$ that contains the same free variables as follows.*

(i) If $A \in \Sigma(\mathbb{C})$, then $A^{\circ} := A$.

(ii) If $A(U, V), B(U, u), D_1(U), D_2(U) \in \mathbb{C}$, then for $j \in \{\wedge, \vee\}$,

- *(a) $(\exists Y \exists X A(X, Y))^{\circ} := \exists X A((X)_0, (X)_1))$,*
- *(b) $(\exists y \exists X B(X, y))^{\circ} := \exists X \exists y B(X, y)$,*
- *(c) $(\exists X D_1(X)\ j\ \exists X D_2(X))^{\circ} := \exists X(D_1((X)_0)\ j\ D_2((X)_1))$.*

(iii) $(A\ j\ B)^\circ := (A^\circ\ j\ B^\circ)^\circ$, $(\exists X A)^\circ := (\exists X(A^\circ))^\circ$, *and* $(\exists x A)^\circ := (\exists x(A^\circ))^\circ$.

Further, if $A \in \mathsf{e}\Pi(\mathbb{C})$, *then* $A^\circ := \neg(\neg A)^\circ$.

Lemma V.4.2. *Assume that* $\mathsf{sufo}^-(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathsf{cut}(\mathbb{D}, \mathsf{T}|\mathsf{S})$. *If* $A \in \mathsf{e}\Sigma(\mathbb{C})$, *then*

$$\overset{*}{\mathsf{T}}|\mathsf{S} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, A \Longrightarrow \overset{*}{\mathsf{T}}|\mathsf{S} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, A^\circ,\ \textit{and}\ \overset{*}{\mathsf{T}}|\mathsf{S} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \neg A \Longrightarrow \overset{*}{\mathsf{T}}|\mathsf{S} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \neg A^\circ.$$

Proof We just show the first claim; the second is shown similarly. Below, we write $A^\circ(U)$ for $(A(U))^\circ$ which is justified as $\mathsf{FV}(A) = \mathsf{FV}(A^\circ)$. Further, we just write $\overset{*}{\mathsf{T}}$ for $\overset{*}{\mathsf{T}}|\mathsf{S}$. The proof is by induction on the definition of A° and side-induction on α. If $A \in \Sigma(\mathbb{C})$, then $A^\circ = A$ and the claim holds trivially. Next, we have a look at the cases (ii)(a)–(ii)(c) of Definition V.4.1. We start with the following auxiliary claims:

(i) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X A(X, U)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists X A((X)_0, (X)_1)$.

(ii) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X A(X, s)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists X \exists x A(X, x)$.

(iii) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X A(X), B(U)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists Y(A((Y)_0) \vee B((Y)_1)$, and
$\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X A(X)$ and $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, B(U)$ imply $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists Y(A((Y)_0) \wedge B((Y)_1)$.

All three claims are shown by induction on α. Exemplarily we show the first one. If $\exists X A(X, U) \in \mathsf{inst}^*(\mathsf{S})$, then as $\overset{*}{\mathsf{T}} \mid\!\frac{<\omega}{*}\ \neg\exists X A(X, U), \exists X A((X)_0, (X)_1)$, the claim follows by a cut. If $\overset{*}{\mathsf{T}} \mid\!\frac{\beta}{\mathbb{D}}\ \Gamma, \exists X A(X, U), A(V, U)$ for a $\beta < \alpha$, then $\overset{*}{\mathsf{T}} \mid\!\frac{<\beta^+}{\mathbb{D}}\ \Gamma, \exists X A((X)_0, (X)_1), A(V, U)$ is obtained by the I.H. Hence, for some fresh Y, Lemma V.3.12 yields

$$\overset{*}{\mathsf{T}} \mid\!\frac{<\beta^+}{\mathbb{D}}\ \Gamma, \exists X A((X)_0, (X)_1), V \neq (Y)_0, U \neq (Y)_1, A((Y)_0, (Y)_1).$$

Since $\overset{*}{\mathsf{T}} \mid\!\frac{<\omega}{*}\ \exists Y[V = (Y)_0 \wedge U = (Y)_1]$, the claim easily follows using Lemma V.3.15. With the above auxiliary claims at hand, one following corresponding claims are readily obtained by induction on α.

(i) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X \exists Y A(X, Y)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists X A((X)_0, (X)_1)$.

(ii) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists x \exists X A(X, x)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists X \exists x A(X, x)$.

(iii) $\overset{*}{\mathsf{T}} \mid\!\frac{\alpha}{\mathbb{D}}\ \Gamma, \exists X D_1(X)\ j\ \exists X D_2(X)$ implies $\overset{*}{\mathsf{T}} \mid\!\frac{<\alpha^+}{\mathbb{D}}\ \Gamma, \exists X(D_1((X)_0)\ j\ D_2((X)_1)$.

Now for the induction step, i.e. case (iii) of Definition V.4.1. We just show the first two cases, the third is shown similarly. If $\overset{*}{\mathsf{T}} \vdash^{\beta+1}_{\mathbb{D}} \Gamma, A$ is obtained form $\overset{*}{\mathsf{T}} \vdash^{\beta}_{\mathbb{D}} \Gamma, A_1, A_2$, then by I.H., $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, A_1^\circ, A_2^\circ$, so $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, A_1^\circ \vee A_2^\circ$, and again by I.H., $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, (A_1^\circ \vee A_2^\circ)^\circ$. Further, if $\overset{*}{\mathsf{T}} \vdash^{\beta+1}_{\mathbb{D}} \Gamma, \exists X A(X)$ is obtained form $\overset{*}{\mathsf{T}} \vdash^{\beta}_{\mathbb{D}} \Gamma, \exists X A(X), A(U)$, then by side I.H., $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, (\exists X A(X))^\circ, A(X)$, and by I.H., $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, (\exists X A(X))^\circ, A^\circ(U)$. Then also $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, (\exists X A(X))^\circ, \exists X A^\circ(X)$, and again by I.H., $\overset{*}{\mathsf{T}} \vdash^{<\beta^+}_{\mathbb{D}} \Gamma, (\exists X A(X))^\circ, (\exists X A^\circ(X))^\circ$. As $(\exists X A(X))^\circ = (\exists X A^\circ(X))^\circ$, this is the claim. □

This allows for the following strengthening of Lemma V.3.15.

Lemma V.4.3. *Let* $\mathsf{sufo}^-(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathbb{C}$ *and* $\mathsf{sufo}^-(A^\circ) \subseteq \mathbb{C}$. *If* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, A$ *and* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\beta}_{\mathbb{C}} \Delta, \neg A$, *then* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<(\alpha \# \beta)^+}_{\mathbb{C}} \Gamma$.

Moreover, it allows us to replace A by a provable consequence B, if $\mathsf{sufo}^-(A^\circ) \subseteq \mathbb{C}$, without a significant increase of the depth of the derivation.

Corollary V.4.4. *Let* $\mathsf{sufo}^-(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathbb{C}$ *and* $\mathsf{sufo}^-(A^\circ) \subseteq \mathbb{C}$. *If* $\mathsf{T}|\mathsf{S} \vdash \neg A, B$ *and* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, A$, *then* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, B$.

Finally, we consider two theories $\mathsf{T}|\mathsf{S}$ and $\mathsf{T}'|\mathsf{S}$, where say, $\mathsf{T}|\mathsf{S} \vdash B$ for each $B \in \mathsf{inst}^*(\mathsf{T}'|\mathsf{S})$. We look for criteria which guarantee that $\overset{*}{\mathsf{T}}'|\mathsf{S} \vdash^{<\gamma}_{\mathbb{C}} \Gamma \Rightarrow \overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\gamma}_{\mathbb{C}} \Gamma$.

Lemma V.4.5. *If for each* $B \in \mathsf{inst}^*(\breve{\mathsf{T}}'|\mathsf{S})$, *there is a* $A \in \mathsf{inst}^*(\breve{\mathsf{T}}|\mathsf{S})$ *so that for each* Γ, $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, \neg B \Rightarrow \overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg A$, *then also for each* Γ, $\overset{*}{\mathsf{T}}'|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma \Rightarrow \overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma$.

Proof By induction on α. We just give the relevant case. If $\overset{*}{\mathsf{T}}'|\mathsf{S} \vdash^{\alpha+1}_{\mathbb{C}} \Gamma$ is obtained from $\overset{*}{\mathsf{T}}'|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, \neg B$ by a cut with some $B \in \mathsf{inst}^*(\breve{\mathsf{T}}'|\mathsf{S})$, then by I.H., $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg B$, and by assumption, $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma, \neg A$ for some $A \in \mathsf{inst}^*(\breve{\mathsf{T}}|\mathsf{S})$, and the claim follows by a cut with A. □

The next lemma describes a situation where this criterion applies.

Lemma V.4.6. *Let* $\mathsf{sufo}^-(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathbb{C}$. *Assume that for each* $B \in \mathsf{inst}^*(\breve{\mathsf{T}}'|\mathsf{S})$, $\mathsf{T}|\mathsf{S} \vdash B$ *and* $\mathsf{sufo}^-(B^\circ) \subseteq \mathbb{C}$. *If* $\overset{*}{\mathsf{T}}'|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma$, *then* $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma$.

Proof The criterion given in Lemma V.4.5 clearly holds: under the given assumptions, $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{\alpha}_{\mathbb{C}} \Gamma, \neg B$ and $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\omega}_{\mathbb{C}} B$ yield $\overset{*}{\mathsf{T}}|\mathsf{S} \vdash^{<\alpha^+}_{\mathbb{C}} \Gamma$ by Lemma V.4.3. □

When we later consider equivalent theories with different axiomatizations, we are in an even better situation. We do not have to cut with an instances of $\mathsf{T}'|\mathsf{S}$, but only with a subformula, since corresponding instances A of $\mathsf{T}|\mathsf{S}$ and B of $\mathsf{T}'|\mathsf{S}$ only differ by some equivalent subformulas A' and B', and it is further the case that if $\overset{*}{\mathsf{T}}|\mathsf{S} \mathrel{\vert\!\frac{<\gamma}{\mathbb{C}}} \Gamma, \neg A$, then we can dig out $\neg A'$ using inversion, replace $\neg A'$ by $\neg B'$ by cutting with $\overset{*}{\mathsf{T}}|\mathsf{S} \mathrel{\vert\!\frac{<\omega}{*}} \neg B', A'$, and then obtain $\overset{*}{\mathsf{T}}|\mathsf{S} \mathrel{\vert\!\frac{<\gamma}{\mathbb{C}}} \Gamma, \neg B$ by undoing the inversion steps.

The following lemma describes such a situation. The idea is that $\neg\forall X A(X)$ is an instance of $\breve{\mathsf{T}}|\mathsf{S}$ and $\neg\forall X B(X)$ the corresponding instance of $\breve{\mathsf{T}}'|\mathsf{S}$. In the lemma below, we could replace the assumption $\mathsf{sufo}^-(A_{i_0}) \subseteq \mathbb{C}$ by $\mathsf{sufo}^-(A^\circ_{i_0}) \subseteq \mathbb{C}$ and $\mathsf{sufo}^-(\exists X[U = (X)_0 \wedge V = (X)_1]) \subseteq \mathbb{C}$.

Lemma V.4.7. *Let I be finite, and $A(U) := \bigvee_{i\in I} A_i(U)$ and $B(U) := \bigvee_{i\in I} B_i(U)$, so that $A_i = B_i$ for $i \in I \setminus \{i_0\}$, and $\mathsf{T}|\mathsf{S} \vdash \neg A_{i_0}, B_{i_0}$ and $\mathsf{sufo}^-(A_{i_0}) \subseteq \mathbb{C}$. Then,*

$$\overset{*}{\mathsf{T}}|\mathsf{S} \mathrel{\vert\!\frac{\alpha}{\mathbb{C}}} \Gamma, \forall X A(X) \;\Rightarrow\; \overset{*}{\mathsf{T}}|\mathsf{S} \mathrel{\vert\!\frac{<\alpha^+}{\mathbb{C}}} \Gamma, \forall X B(X).$$

Chapter VI

Finitary and infinitary reductions and sharp bounds

Our aim is to show that for $g(\alpha) := \omega^{1+\alpha}$, $\mathsf{T} := \mathsf{ACA}_0$ and $\mathsf{T}_x := \mathsf{Op}_x(\mathsf{T})$, g_{x^h} is indeed that largest normal function which is provable in T_x, by which me mean that for any other normal function that is provable in T_x, we have that $f{\restriction}\mathsf{Lim}(\Omega) \leq g_{x^h}{\restriction}\mathsf{Lim}(\Omega)$. We call such a largest provable function a *sharp bound* of T_x.

To show that g_{x^h} is a sharp bound of T_x, we employ the more general notion of a bound f of T_x, that is, a normal function, so that for each name $x \in Q$, each derivation $\overset{*}{\mathsf{T}}_x|\mathsf{S} \vdash^{<\gamma}_{+} \Gamma$ of an arithmetical $\mathcal{L}^*(\mathsf{T}|\mathsf{S})$-sequent Γ can be transformed into a derivation $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{S} \vdash^{<f(\gamma)}_{*} \Gamma$, where $\overset{*}{\mathsf{T}}_x|\mathsf{S} \vdash^{<\gamma}_{+} \Gamma$ indicates that the cut-rule is restricted to formulas in $\mathsf{inst}^*(\breve{\mathsf{T}}_x|\mathsf{S})$ and some additional formulas that do not impede the cut-elimination process; these additional cuts can be eliminated cheaply at a later stage. We show that g_{x^h} is a bound of T_x. Using the Boundedness Lemma, we obtain that g_{x^h} is also a sharp bound.

We start this chapter by reviewing the axiomatizations of the theories $(\mathsf{T}_x : x \in Q^*)$. Then, we are ready to prove the reduction properties listed below. Thereby, we make use of the approximations $x[\alpha]$ and $x(\alpha)$ (cf. Definition III.4.10). First, we look at the following reductions which are feasible without resorting to infinitary systems.

(i) If $x \in Q$, and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$, then for some n,

$$\mathsf{T}_{x+1}|\mathsf{S} \vdash \Gamma \;\Rightarrow\; \mathsf{T}_x^{n+1}|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma.$$

Note that if Γ is arithmetical and $\mathsf{FV}_1(\Gamma) = \emptyset$, then $\mathsf{M}_1 \models \Gamma$ is Γ.

(ii) If $x \in Q$ with $\deg(x) = m{+}2$, and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$, then for some n,

$$\mathsf{T}_x|\mathsf{S} \vdash \Gamma \;\Rightarrow\; \mathsf{T}_{x(n)}|\mathsf{S} \vdash \Gamma.$$

Next, we look at the related infinitary reductions.

(iii) If $x \in Q$, f is a bound of T_x and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$, then

$$\overset{*}{\mathsf{T}}_{x+1}|\mathsf{S} \,\Big|\!\frac{\ <\gamma\ }{\mathcal{L}^*_{\Pi^1_0}}\, \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}_{x}|\mathsf{S} \,\Big|\!\frac{\ <\mathsf{it}(f,\gamma)\ }{*}\, \Gamma.$$

(iv) If $x \in Q$ with $\deg(x) = m{+}2$, and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$, then

$$\overset{*}{\mathsf{T}}_{x}|\mathsf{S} \,\Big|\!\frac{\ \alpha\ }{\mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}}\, \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}_{x(\alpha)}|\mathsf{S} \,\Big|\!\frac{\ <\alpha^+\ }{\mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}}\, \Gamma.$$

In order to obtain (iii), we use that if f is a bound of T_x and say $\overset{*}{\mathsf{T}}{}^2_x|\mathsf{S} \,\Big|\!\frac{\ <\gamma\ }{+}\, \mathsf{M}_1 \models \Gamma$, then $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_x|\mathsf{S} \,\Big|\!\frac{\ <f(\gamma)\ }{*}\, \mathsf{M}_1 \models \Gamma$, which in turn yields $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\Big|\!\frac{\ <f(\gamma)\ }{*}\, \Gamma$ by Lemma VI.3.13, an important auxiliary result, that we show prior to the above reduction properties (iii)–(iv). With (iii)–(iv) at hand, we are ready to produce a first proof that for each $x \in Q$, g_{x^h} is a bound of T_x.

In the final section, we then show, dually to what we did in Section III.7, that in some higher type sense, H_{x^H} is a bound of Op_x, and that $H^{+(n+1)}_{x^*}$ is a bound of $\mathsf{Op}^{+(n+1)}_x$ (recall that $x^* := x^H{+}\mathsf{corr}(x)$; see Definitions III.4.20). Finally, we conclude by discussing what meta-theory we implicitly used to prove these results.

VI.1 Revisiting the axioms of $(\mathsf{T}_x : x \in Q^*)$

When transforming a derivation of one theory into a derivation of another, clearly the exact form of the axioms $\breve{\mathsf{T}}_x$ of the involved theories T_x matter. However, whenever possible, we avoid working with the rather complicated sentence $\breve{\mathsf{T}}_x$, but work instead, if say $\deg(x) = m{+}1$, with the axiom $\breve{\mathsf{T}}'_x := (\forall \alpha \lhd o(x))\mathsf{p}_{m+1}(\breve{\mathsf{T}}_{x[\alpha]})$ of the theory $\mathsf{T}'_x := \mathsf{T}^{\epsilon} + \breve{\mathsf{T}}'_x$, which by Lemma III.6.2 (iii) proves the same formulas. Although the axiom $(\forall \alpha \lhd o(x))\mathsf{p}^{+n}_{m+1}(\breve{\mathsf{T}}_{x[\alpha]})$ contains $\breve{\mathsf{T}}_{x[\alpha]}$ as a subformula, we can mostly avoid looking inside $\breve{\mathsf{T}}_{x[\alpha]}$.

If we are concerned with finitary reductions where provability is all that matters, looking at T'_x instead of T_x causes no problems. However, when considering infinitary reductions, also the derivation's depth and the complexity of its cut-formulas become relevant. Therefore, switching from $\overset{*}{\mathsf{T}}_x$ to $\overset{*}{\mathsf{T}}{}'_x$ needs some justification, which involves inspecting the non-arithmetical $\mathcal{L}^*$-instances of T_x and T'_x, respectively.

This inspection reveals (see Lemma VI.1.2 below) that if $\deg(x) = m{+}1$, then for each $A \in \mathsf{inst}^*(\breve{\mathsf{T}}_x|\mathsf{S})$ and each $B \in \mathsf{inst}^*(\breve{\mathsf{T}}_x|\mathsf{S})$, we have $A, B \in \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}$, and therefore, $\mathsf{sufo}^-(A^\circ) \subseteq \mathcal{L}^*_{\mathsf{e}\Pi^1_m}$ and $\mathsf{sufo}^-(B^\circ) \subseteq \mathcal{L}^*_{\mathsf{e}\Pi^1_m}$. As further, $\mathsf{T}_x|\mathsf{S} \vdash B$ and $\mathsf{T}'_x|\mathsf{S} \vdash A$, Lemma V.4.6 yields the following.

Lemma VI.1.1. *Let* $\deg(x) = m{+}1$ *and* $\breve{\mathsf{T}}'_x := (\forall\alpha < o(x))\mathsf{p}_{m+1}(\overset{*}{\mathsf{T}}_{x[\alpha]})$. *Then,*

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \mathrel{\vert\!\frac{<\alpha^+}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_m}}} \Gamma \Leftrightarrow \overset{*}{\mathsf{T}}'_x|\mathsf{S} \mathrel{\vert\!\frac{<\alpha^+}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_m}}} \Gamma.$$

Now we inspect the $\mathcal{L}^*$-instances of T_x and T'_x. Recall that $\breve{\mathsf{T}}_x = \varphi(x)\{\breve{\mathsf{T}}{\restriction}U\}$, where $\varphi(x)_{\breve{\mathsf{T}}\restriction U} := \varphi(x)\{\breve{\mathsf{T}}{\restriction}U\}$ is obtained from the $\mathcal{L}(\mathsf{P})$-formula $\varphi(x)$ by replacing each occurrence of $\mathsf{P}(\mathcal{X})$ by $\breve{\mathsf{T}}{\restriction}\mathcal{X}$, and $\varphi(u)$, specified by Definition III.6.1 and Theorem A.1.2, is defined by means of the $\mathcal{L}(\mathsf{P})$-formulas $\psi(X,u) = \exists Y\psi'(X,Y,u)$, where $\psi'(X,Y,u)$ has no set quantifiers, and $\vartheta^{N_0}(u) = u \leq N_0 \wedge \bigwedge_{n<N_0}(u = n{+}1 \to \varphi_{\mathsf{p}_{n+1}})$. N_0 is an arbitrary large but fixed bound on the level of the names. Further,

(i) $\varphi_{\mathsf{p}_1} := \forall Z\exists X[Z\dot{\in}X \wedge \mathsf{P}(X)] \wedge \mathsf{pair} \wedge \mathsf{trans}$, and

(ii) $\varphi_{\mathsf{p}_{n+2}} := \forall Z\forall x,e\exists X[Z\dot{\in}X \wedge \mathsf{P}(X) \wedge R_{n+2}(X,Z,x,e)] \wedge \mathsf{pair} \wedge \mathsf{trans}$.

Since $\varphi(u) = q_0 \rightsquigarrow^* u \wedge \mathsf{Wo}_{\rightsquigarrow^*}(u) \wedge \mathsf{good}(\rightsquigarrow,\prec) \wedge (\forall y \rightsquigarrow u)\vartheta(\deg(u))\{\psi(X,y)\}$,

$$\breve{\mathsf{T}}_x = q_0 \rightsquigarrow^* u \wedge \mathsf{Wo}_{\rightsquigarrow^*}(u) \wedge \mathsf{good}(\rightsquigarrow,\prec) \wedge (\forall y \rightsquigarrow u)\vartheta(\deg(u))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,y)\}.$$

Now we let x be a closed term and observe how the $\mathcal{L}^*$-instances of T_x look like. Recall that $\mathsf{good}(\rightsquigarrow,\prec)$ is an arithmetical sentence that asserts that $\prec$ is the transitive closure of $\rightsquigarrow$ (cf. Definition I.2.24), and note that the $\mathcal{L}^*$-instances of $\mathsf{Wo}_{\rightsquigarrow^*}(x)$ are arithmetical. Next, we unwind $(\forall y \rightsquigarrow x)\vartheta(\deg(x))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,y)\}$, which yields

$$(\forall y \rightsquigarrow x)[\deg(x) \leq N_0 \wedge \bigwedge_{n<N_0}(\deg(x) = n{+}1 \to \varphi_{\mathsf{p}_{n+1}}) \wedge \mathsf{pair} \wedge \mathsf{trans}]\{\psi_{\breve{\mathsf{T}}\restriction U}(X,y)\}.$$

At this point it comes in handy that $\mathsf{inst}^*(A \to B) = \mathsf{inst}^*(B)$ if A is a true literal, and $\mathsf{inst}^*(A \to B) = \emptyset$ if A is a false literal. We see that the non-arithmetical $\mathcal{L}^*$-instances of $\varphi(x)_{\breve{\mathsf{T}}\restriction U}$ (those different from the $\mathcal{L}^*$-instances of $\mathsf{pair} \wedge \mathsf{trans}$) are of the following forms, where y is a closed terms so that $y \rightsquigarrow x$, i.e. $y = x[\alpha]$ for some $\alpha < o(x)$:

(iii) $\exists X[Z\dot{\in}X \wedge \psi_{\breve{\mathsf{T}}\restriction U}(X,y)]$, if $\deg(x) = 1$,

(iv) $\exists X[Z\dot{\in}X \wedge \psi_{\breve{\mathsf{T}}\restriction U}(X,y) \wedge R_{n+2}(X,Z,s,t)]$, if $\deg(x) = n{+}2$.

Since $\psi_{\breve{\mathsf{T}}\restriction U}(X,y)$ is a Σ^1_1-formula and $R_{n+2}(X,Z,s,t) \in \mathcal{L}^*_{\mathsf{e}\Sigma^1_{n+2}}$, each $\mathcal{L}^*$-instance of T_x is in $\mathcal{L}^*_{\mathsf{e}\Sigma^1_{n+2}}$. The same holds for the $\mathcal{L}^*$-instance of $\mathsf{T}'_x|\mathsf{S}$: as $\mathsf{p}_{n+1}(\breve{\mathsf{T}}_{x[\alpha]})$ is $\varphi_{\mathsf{p}_{n+1}}\{\breve{\mathsf{T}}_{x[\alpha]}{\restriction}U\}$, we see that the corresponding $\mathcal{L}^*$-instances of T_x and T'_x possibly only differ by the subformulas $\psi_{\breve{\mathsf{T}}\restriction U}(X,x[\alpha])$ and $\overset{*}{\mathsf{T}}_{x[\alpha]}{\restriction}X$. Therefore, the assumptions of Lemma VI.1.1 are clearly justified.

Lemma VI.1.2. *Let* $\deg(x) = n{+}1$ *and* $\breve{\mathsf{T}}'_x := (\forall\alpha < o(x))\mathsf{p}_{n+1}(\overset{*}{\mathsf{T}}_{x[\alpha]})$. *Then,*

$$\mathsf{inst}^*(\breve{\mathsf{T}}) \cup \mathsf{inst}^*(\breve{\mathsf{T}}') \subseteq \mathcal{L}^*_{\mathsf{e}\Sigma^1_{n+1}}.$$

However, we can do better. Since for each $x \in Q^*$, $\mathsf{T}_x \vdash \mathsf{p}_1((\mathsf{ACA}))$, we have by Lemma I.A.1.13, that

$$(*) \quad \mathsf{T}_x \vdash \psi_{\breve{\mathsf{T}}\restriction U}(X,y) \leftrightarrow \breve{\mathsf{T}}_y\restriction X, \quad \text{and therefore also} \quad \mathsf{T}'_x \vdash \psi_{\breve{\mathsf{T}}\restriction U}(X,y) \leftrightarrow \breve{\mathsf{T}}_y\restriction X.$$

Hence, $(*)$ together with Lemma V.4.5 and V.4.7 yields the following sharper result.

Lemma VI.1.3. *Let* $\deg(x) = n{+}1$, *and* $\breve{\mathsf{T}}'_x := (\forall\alpha < o(x))\mathsf{p}_{n+1}(\overset{*}{\mathsf{T}}_{x[\alpha]})$. *Then,*

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \mathrel{\mathop{|}}\!\frac{<\alpha^+}{\mathcal{L}^*_{\Pi^1_0}}\ \Gamma \Leftrightarrow \overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mathrel{\mathop{|}}\!\frac{<\alpha^+}{\mathcal{L}^*_{\Pi^1_0}}\ \Gamma.$$

VI.2 Finitary reductions

Now we are prepared to perform the finitary reductions announced at the beginning of this chapter. Many of these results are proved by induction on the depth of the derivation. Thereby, the auxiliary result below is of good use.

Lemma VI.2.1. *If for some* n *and each* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Sigma^1_m}$, $\mathsf{S} \mathrel{|}\!\frac{n}{*}\ \Gamma \Rightarrow \mathsf{S}' \vdash \Gamma$, *then also for each* $\Delta \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Pi^1_{m+1}}$, $\mathsf{S} \mathrel{|}\!\frac{n}{*}\ \Delta \Rightarrow \mathsf{S}' \vdash \Delta$.

Proof By induction on $\Sigma_{A\in\Delta}\mathsf{rk}(A)$, the sum of the ranks of the formulas in Δ. If $\Delta \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Sigma^1_m}$, the claim is by assumption. Hence, assume that $\Delta \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Pi^1_{m+1}}$ is of the form Δ', A for $A \in (\mathcal{L}_{\mathsf{e}\Pi^1_{m+1}} \setminus \mathcal{L}_{\mathsf{e}\Sigma^1_m})$. Then A is of the form $\forall y B(y)$, $\forall Y B(Y)$, $B_1 \vee B_2$ or $B_1 \wedge B_2$, where $B(U), B(u), B_1$ and B_2 are still in $\mathcal{L}_{\mathsf{e}\Pi^1_{m+1}}$. If e.g. A is of the form $\forall X B(X)$, then by inversion, $\mathsf{S} \mathrel{|}\!\frac{n}{*}\ \Delta', B(X)$, hence by I.H., $\mathsf{S}' \vdash \Delta', B(X)$, and $\mathsf{S}' \vdash \Delta', A$ follows. And if e.g. A is $B_1 \wedge B_2$, then by inversion, for $i \in \{1,2\}$, $\mathsf{S} \mathrel{|}\!\frac{n}{*}\ \Delta', B_i$, hence by I.H., $\mathsf{S}' \vdash \Delta', B_i$, and $\mathsf{S}' \vdash \Delta', A$ follows. The cases where A is $\forall y B(y)$ or $B_1 \vee B_2$ is handled similarly. □

VI.2.1 Reducing T_{x+1} to T^n_x

The $\mathcal{L}$-instances of $\mathsf{p}_1(\mathsf{T})|\mathsf{S}$ consist of the $\mathcal{L}$-instances of $\breve{\mathsf{S}}\restriction\mathsf{M}_0$, $\exists X[X = \mathsf{M}_0]$, pair and trans (cf. Definition I.2.3), which are also among the $\mathcal{L}$-instances of $\mathsf{T}|\mathsf{S}$, and $\mathcal{L}$-instances of the form $A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z)$, called the relevant $\mathcal{L}$-instances, defined below.

Definition VI.2.2. *For each set variable* Z, $A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z) := \exists X[Z \dot{\in} X \wedge \breve{\mathsf{T}}\restriction X]$ *is called a relevant instance of* $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S}$.

Next, we show a first auxiliary reduction property.

Lemma VI.2.3. *Let* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$ *and* $n > 0$. *Then,*

$$\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \mathsf{M}_1 \models \Gamma \;\Rightarrow\; \mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma.$$

Proof Assume that $\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \mathsf{M}_1 \models \Gamma$, where we suppose that $\mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z)$ is $\forall x[(\mathsf{M}_1)_z \dot{\notin} (\mathsf{M}_1)_x \vee \neg\breve{\mathsf{T}}{\restriction}(\mathsf{M}_1)_x]$. Since M_1 is a transitive set, we have that

$$(*) \qquad \mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash Z \dot{\notin} X, X \dot{\notin} \mathsf{M}_1, \neg\breve{\mathsf{T}}{\restriction}X, (\mathsf{M}_1 \models \Gamma)[Z/(\mathsf{M}_1)_z]).$$

Using Lemma V.2.8 (iii), we obtain that

$$(*) \qquad \mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \forall X(Z \dot{\in} X \wedge X \dot{\in} \mathsf{M}_2 \wedge \breve{\mathsf{T}}{\restriction}X \to (\mathsf{M}_2 \models \Gamma)[Z/(\mathsf{M}_2)_z]).$$

Now we work informally in $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S}$. Assume $(*)$. As M_1 is a set with $\mathsf{M}_1 \dot{\in} \mathsf{M}_2$ and $\breve{\mathsf{T}}{\restriction}\mathsf{M}_1$, we can instantiate X with M_1, and obtain that $Z \dot{\in} \mathsf{M}_1 \to (\mathsf{M}_2 \models \Gamma)[Z/(\mathsf{M}_2)_z]$. As (Γ) is arithmetical and $\mathsf{M}_1 \dot{\subseteq} \mathsf{M}_2$, we also have $Z \dot{\in} \mathsf{M}_1 \to (\mathsf{M}_1 \models \Gamma)[Z/(\mathsf{M}_1)_z]$. Instantiating Z with $(\mathsf{M}_1)_z$ finally yields $\mathsf{M}_1 \models \Gamma$. $\Box$

Lemma VI.2.4. *Let* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$. *Then,*

$$\mathsf{p}_1(\mathsf{T})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, \Gamma \;\Rightarrow\; \mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma.$$

Proof By induction on n. If $n = 0$, then Γ is an axiom of $\mathsf{p}_1(\mathsf{T})|\mathsf{S}$. As Γ is arithmetical, its main formula is an instance of $\breve{\mathsf{S}}{\restriction}\mathsf{M}_0$ or T^ϵ. Thus, Γ is already an axiom of $\mathsf{T}^\epsilon|\mathsf{S}$. Hence $\mathsf{T}^\epsilon|\mathsf{S} \vdash \Gamma$, and by Lemma I.1.9, $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma$.
The induction step is immediate from the I.H., except if Γ is obtained by a cut with a non-arithmetical instance A of $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S}$ which is not an instance of pair or trans. Hence, assume that Γ was obtained from $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, \Gamma, \neg A$ by a cut, where A is either $A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z)$ or $\exists X[X = \mathsf{M}_0]$. In both cases, we can assume that $n > 0$ (else, already Γ is an axiom). In the first case, we obtain from $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, \Gamma, \neg A$ by $\forall X$-inversion, that $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, \Gamma, U \neq \mathsf{M}_0$ for $U \notin \mathsf{FV}(\Gamma)$. Now the I.H. applies and yields $\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma, (\mathsf{M}_1)_u \neq \mathsf{M}_0$, where u is a fresh variable. Recall that $\forall x[(\mathsf{M}_1)_x \neq \mathsf{M}_0]$ is $\mathsf{M}_0 \dot{\notin} \mathsf{M}_1$. As $n > 0$, $\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S}$ proves that $\mathsf{M}_0 \dot{\in} \mathsf{M}_1$, therefore $\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma$ follows. Hence, $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \mathsf{M}_2 \models \Gamma$ by Lemma V.2.8 (iii). As Γ is arithmetical and $\mathsf{M}_1 \dot{\subseteq} \mathsf{M}_2$, we also have $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma$. And if Γ was obtained from $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \Gamma$ by a cut, then by $\forall X$-inversion we also have $\mathsf{p}_1(\breve{\mathsf{T}})|\mathsf{S} \,{\overset{n}{\underset{*}{\vdash}}}\, Z \dot{\notin} X \vee \neg\breve{\mathsf{T}}{\restriction}X, \Gamma$ for $X \notin \mathsf{FV}_1(\Gamma)$. Hence the I.H. applies and yields $\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models Z \dot{\notin} X \vee \neg\breve{\mathsf{T}}{\restriction}X, \mathsf{M}_1 \models \Gamma$, from which we further conclude

$$\mathsf{T}^\epsilon|\mathsf{T}^n|\mathsf{S} \vdash \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \mathsf{M}_1 \models \Gamma.$$

Now $\mathsf{T}^\epsilon|\mathsf{T}^{n+1}|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma$ follows by Lemma VI.2.3. $\square$

By Lemma III.6.2 (iii), $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}}_{x+1} \leftrightarrow \mathsf{p}_1(\breve{\mathsf{T}}_x)$. Hence $\mathsf{T}_{x+1}|\mathsf{S}$ and $\mathsf{p}_1(\mathsf{T}_x)|\mathsf{S}$ prove the same formulas. Thus, we the following corollary.

Corollary VI.2.5. *Let* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$. *Then,*

$$\mathsf{T}_{x+1}|\mathsf{S} \mathrel{|\!\!\frac{n}{*}} \Gamma \;\Rightarrow\; \mathsf{T}^\epsilon|\mathsf{T}^n_x|\mathsf{S} \vdash \mathsf{M}_1 \models \Gamma.$$

VI.2.2 Reducing T_x to $\mathsf{T}_{x(n)}$ for names $x \in Q$ with $\deg(x) > 1$

If $\deg(x) = m{+}2$, then $\mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$ iff $(\forall\alpha \lhd o(x))(\mathsf{p}_{m+n+2}\mathsf{Op}^{+n}_{x[\alpha]}(\breve{\mathsf{T}}))$ (cf. Lemma III.6.2 (iii)). Also recall that

$$\mathsf{p}_{n+2}(\breve{\mathsf{T}}) = \forall Z \forall u, v \exists X(Z \dot{\in} X \wedge \breve{\mathsf{T}}{\restriction}X \wedge R_{n+2}(X, Z, u, v)) \wedge \mathsf{pair} \wedge \mathsf{trans},$$

where $R_{n+2}(X, Z, u, v) = \pi^1_{n+2}(Z, u, v) \to \pi^1_{n+2}(Z, u, v){\restriction}X$ (cf. Definition I.I.2.8). For each T and each $x \in Q$ with $\deg(x) = m{+}2$ and $n \in \mathbb{N}$, we let

$$A_{\breve{\mathsf{T}},x,n}(Z, \alpha, s, t) := \alpha \lhd o(x) \to \exists X(Z \dot{\in} X \wedge \mathsf{Op}^{+n}_{x[\alpha]}(\breve{\mathsf{T}}){\restriction}X \wedge R_{m+n+2}(X, Z, s, t)).$$

Then, $\mathsf{T}^\epsilon \vdash \forall Z, \alpha, u, v A_{\breve{\mathsf{T}},x,n}(Z, \alpha, u, v) \wedge \mathsf{pair} \wedge \mathsf{trans} \leftrightarrow \mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$ by Lemma III.6.2 (iii), and for all number terms α, s, t, $A_{\breve{\mathsf{T}},x,n}(Z, \alpha, s, t) \in \mathcal{L}_{\mathsf{e}\Sigma^1_{m+n+2}}$ is a relevant $\mathcal{L}$-instance of $\breve{\mathsf{T}}'^{+n}_x := (\forall\alpha \lhd o(x))\mathsf{p}_{m+n+2}(\breve{\mathsf{T}}^{+n}_{x[\alpha]})$, one that is not also an instance of some $\breve{\mathsf{T}}^{+n}_{x(\beta)}$.

The following simple observation will be employed in the proof of Lemma VI.2.9, the key lemma of this subsection.

Lemma VI.2.6. *If* $\mathsf{T}|\mathsf{S} \vdash \Gamma$ *and* $Y \notin \mathsf{FV}_1(\Gamma) = \{Z_1, \ldots, Z_k\}$, *then*

$$\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \mathrel{\dot{\notin}} Y, \vec{Z} \mathrel{\dot{\notin}} Y, \neg\breve{\mathsf{T}}{\restriction}Y, \Gamma{\restriction}Y.$$

Proof If $\mathsf{T}|\mathsf{S} \vdash \Gamma$, then also $\mathsf{T}^\epsilon \vdash \Delta$ for $\Delta := \neg(\breve{\mathsf{S}}{\restriction}\mathsf{M}_0), \neg\exists X[X = \mathsf{M}_0], \neg\breve{\mathsf{T}}, \Gamma$, and so $\mathsf{T}^\epsilon \vdash Y \models \Delta$ by Lemma V.2.7. As $\mathsf{T}^\epsilon \vdash (\neg\exists X[X = \mathsf{M}_0]){\restriction}Y \leftrightarrow \mathsf{M}_0 \mathrel{\dot{\notin}} Y$, and $\neg(\breve{\mathsf{S}}{\restriction}\mathsf{M}_0)$ is arithmetical without free set variables, $\mathsf{T}^\epsilon \vdash \neg(\breve{\mathsf{S}}{\restriction}\mathsf{M}_0), \mathsf{M}_0 \mathrel{\dot{\notin}} Y, \vec{Z} \mathrel{\dot{\notin}} Y, \neg\breve{\mathsf{T}}{\restriction}Y, \Gamma{\restriction}Y$ follows from $\mathsf{T}^\epsilon \vdash \Delta$. Hence, $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \mathrel{\dot{\notin}} Y, \vec{Z} \mathrel{\dot{\notin}} Y, \neg\breve{\mathsf{T}}{\restriction}Y, \Gamma{\restriction}Y$. $\square$

Next, we review some properties concerning the interplay of names, normal forms and operations. If $\deg(x) = m{+}2$ and $x =_{NF} y \circ_m z$, then y is a simple name of degree two and thus of the form $(1, y')$ with $\deg(y') = 1$ (cf. Definition III.4.6). Further, $o(y) = o(x)$ (cf. Definition III.4.3 (ii) and Lemma III.4.2 (ii), (iii)). Moreover, we have the following.

Lemma VI.2.7. *Let y be a simple name with* $\deg(y) = 2$.

(i) *If* $o(y) = 1$, *then* $y(\beta{+}1)[0] = y[0] \circ y[\beta]$ *and* $y(0)[0] = y[0]$.

(ii) *If* $o(y) \in \mathsf{Lim}(\Omega)$ *and* $\alpha \lhd o(y)$, *then* $y(\beta{+}1)[\alpha] = y[\alpha] \circ y(\beta)$ *and* $y(0)[\alpha] = y[\alpha]$.

Proof As y is a simple name of degree two, we have that $y = (1, z)$, where $\deg(z) = 1$. (i) If $o(y) = 1$, then $y(\delta)[0] = (y[\delta]{+}1)[0] = y[\delta]$. Further, $y[\beta{+}1] = y[0] \circ y[\beta]$ by Lemma III.4.13. The claim follows. (ii) If $o(y) \in \mathsf{Lim}(\Omega)$, then $z^- \in P$ and $o(z) = o(y)$ (cf. Lemma III.4.2 (i)). Then, $y(0) = (1, z^-)$ and $y(0)[\alpha] = (1, z[\alpha]) = y[\alpha]$ by Definition III.4.10, and $y(\beta{+}1) = (1, z^-) \circ (1{+}\beta, z^-) = y(0) \circ y(\beta)$. So $y(\beta{+}1)[\alpha] = y(0)[\alpha] \circ y(\beta) = y[\alpha] \circ y(\beta)$. $\square$

The next lemma provides some auxiliary properties of operations. Recall that for each operation Op, $\mathsf{T}^\epsilon \vdash \breve{\mathsf{T}} \to \breve{\mathsf{T}}'$ implies $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}(\breve{\mathsf{T}}')$. Further, $\mathsf{Op} \Rightarrow \mathsf{Op}'$ states that for each T, $\mathsf{T}^\epsilon \vdash \mathsf{Op}(\breve{\mathsf{T}}) \to \mathsf{Op}'(\breve{\mathsf{T}})$. Moreover, (cf. Definition III.6.1), $\breve{\mathsf{T}}^{+n}_x := (x = q_0 \wedge \breve{\mathsf{T}}) \vee (x \neq q_0 \wedge \mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$. Thus, we have that for $x \neq q_0$, $\breve{\mathsf{T}}^{+n}_x \leftrightarrow \mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$, and we can use $\breve{\mathsf{T}}^{+n}_x$ as a more compact way to write $\mathsf{Op}^{+n}_x(\breve{\mathsf{T}})$.

Lemma VI.2.8. *Assume that* $\breve{\mathsf{T}}$ *is* Π^1_{n+2}. *The following is provable in* T^ϵ. *If* $z_0, z_1 \in Q^*$ *and y is a simple name of degree two, then*

(a) $\mathsf{Op}^{+n}_{z_1}(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}$, $(\mathsf{Op}^{+n}_{z_0} \circ \mathsf{Op}^{+n}_{z_1})(\breve{\mathsf{T}}) \to \mathsf{Op}^{+n}_{z_0}(\breve{\mathsf{T}})$, *and if* $\deg(z_1) = 1$, *then* $\mathsf{Op}^{+n}_{z_0} \circ \mathsf{Op}^{+n}_{z_1} \Rightarrow \mathsf{Op}^{+n}_{z_1}$.

(b) $\alpha \lhd o(y) \wedge \breve{\mathsf{T}}^{+n}_{y(\beta+1)} \to \mathsf{p}_{n+1}(\breve{\mathsf{T}}^{+n}_{y[\alpha]} \wedge \breve{\mathsf{T}}^{+n}_{y(\beta)})$, *and* $\alpha \lhd o(y) \wedge \breve{\mathsf{T}}^{+n}_{y(0)} \to \mathsf{p}_{n+1}(\breve{\mathsf{T}}^{+n}_{y[\alpha]})$.

Proof (a) By Lemma III.6.5 (i), $\mathsf{Op}^{+n}_{z_1} \Rightarrow \mathsf{p}_{n+1}$, and as $\breve{\mathsf{T}}$ is Π^1_{n+2}, Lemma I.2.12 yields $\mathsf{T}^\epsilon \vdash \mathsf{p}_{n+1}(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}$. So $\mathsf{T}^\epsilon \vdash \mathsf{Op}^{+n}_{z_1}(\breve{\mathsf{T}}) \to \breve{\mathsf{T}}$, thus $\mathsf{T}^\epsilon \vdash (\mathsf{Op}^{+n}_{z_0} \circ \mathsf{Op}^{+n}_{z_1})(\breve{\mathsf{T}}) \to \mathsf{Op}^{+n}_{z_0}(\breve{\mathsf{T}})$ follows. And if $\deg(z_1) = 1$, then by Lemma III.6.4 (ii), $\mathsf{Op}^{+n}_{z_1}(\breve{\mathsf{T}})$ is Π^1_{n+2}, hence for each $\breve{\mathsf{T}}'$, $(\mathsf{Op}^{+n}_{z_0} \circ \mathsf{Op}^{+n}_{z_1})(\breve{\mathsf{T}}') \to \mathsf{Op}^{+n}_{z_1}(\breve{\mathsf{T}}')$ by the first claim.

(b) We just show the first claim, the second is shown similarly but simpler. First, we show that $\mathsf{Op}^{+n}_{y(\beta+1)} \Rightarrow \mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{y[\alpha]} \circ \mathsf{Op}^{+n}_{y(\beta)})$ for each $\alpha < \gamma$, and then that $(\mathsf{Op}^{+n}_{y[\alpha]} \circ \mathsf{Op}^{+n}_{y(\beta)})(\breve{\mathsf{T}})$ implies $\mathsf{Op}^{+n}_{y[\alpha]}(\breve{\mathsf{T}})$ and $\mathsf{Op}^{+n}_{y(\beta)}(\breve{\mathsf{T}})$. This yields the claim.

If $o(y) = 1$, then $y(\beta) = y[\beta]{+}1$. $\mathsf{Op}^{+n}_{y(\beta+1)} \Leftrightarrow \mathsf{p}_{n+1}\mathsf{Op}^{+n}_{y(\beta+1)[0]} \Leftrightarrow \mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{y[0]} \circ \mathsf{Op}^{+n}_{y[\beta]})$ is by Lemma VI.2.7 (i). By (a) we obtain $(\mathsf{Op}^{+n}_{y[0]} \circ \mathsf{Op}^{+n}_{y[\beta]})(\breve{\mathsf{T}}) \to \mathsf{Op}^{+n}_{y[0]}(\breve{\mathsf{T}})$, and further $\mathsf{Op}^{+n}_{y[0]} \circ \mathsf{Op}^{+n}_{y[\beta]} \Rightarrow \mathsf{p}_{n+1} \circ \mathsf{Op}^{+n}_{y[\beta]} \Rightarrow \mathsf{Op}^{+n}_{y(\beta)}$. And if $o(y) \in \mathsf{Lim}(\Omega)$ and $\alpha \lhd o(y)$, then $\mathsf{Op}^{+n}_{y(\beta+1)} \Rightarrow \mathsf{p}_{n+1}(\mathsf{Op}^{+n}_{y[\alpha]} \circ \mathsf{Op}^{+n}_{y(\beta)})$ by Lemma VI.2.7 (ii). Since $\deg(y(\beta)) = 1$ by Lemma III.4.11, $(\mathsf{Op}^{+n}_{y[\alpha]} \circ \mathsf{Op}^{+n}_{y(\beta)})(\breve{\mathsf{T}}) \to \mathsf{Op}^{+n}_{y[\alpha]}(\breve{\mathsf{T}}) \wedge \mathsf{Op}^{+n}_{y(\beta)}(\breve{\mathsf{T}})$ is by (a). $\square$

There is one more thing we wish to recall. If $\deg(y) = 2$ and $\beta \lhd \alpha$, then by Lemma III.4.19 (ix), $y(\beta) \rightsquigarrow^* y(\alpha)$, and so by Lemma III.6.5 (i), $\mathsf{T}^{+m}_{y(\beta)} \vdash \mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+m}_{y(\alpha)})$, and as $\deg(y(\alpha)) = 1$ further $\mathsf{T}^\epsilon \vdash \mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+m}_{y(\alpha)}) \to \breve{\mathsf{T}}^{+m}_{y(\alpha)}$ (e.g. by Lemma VI.2.8 (a)). So $\mathsf{T}^{+m}_{y(\beta)}|\mathsf{S} \vdash \breve{\mathsf{T}}^{+m}_{y(\alpha)}|\mathsf{S}$.

After these preparatory steps, the following reduction property is readily proved.

Lemma VI.2.9. *Let y be a simple name of degree two. Further, assume that $\breve{\mathsf{T}}$ is Π^1_{m+2} and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$. Then,*

$$\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \Gamma, \neg A_{\breve{\mathsf{T}},y,m}(Z,\alpha,s,t) \;\Rightarrow\; \mathsf{T}^{+m}_{y(n+1)}|\mathsf{S} \vdash \Gamma.$$

Proof Suppose that $\breve{\mathsf{T}}$, Γ and y meet the assumptions of the lemma and that further $\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \Gamma, \neg A_{\breve{\mathsf{T}},y,m}(Z,\alpha,s,t)$, where $\mathsf{FV}_1(\Gamma) \cup \{Z\} = \{Z_1,\ldots,Z_k\}$. As the formula $\neg A_{\breve{\mathsf{T}},y,m}(Z,\alpha,s,t)$ looks as follows,

$$\alpha \lhd o(y) \wedge \forall X[Z \mathrel{\dot{\notin}} X \vee \neg(\breve{\mathsf{T}})^{+m}_{y[\alpha]}{\restriction}X \vee (\pi^1_{m+2}(Z,s,t) \wedge \neg\pi^1_{m+2}(Z,s,t){\restriction}X)],$$

we obtain for $X, Y \notin \mathsf{FV}_1(\Gamma)$, using inversion,

(i) $\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \Gamma, \alpha \lhd o(y)$,

(ii) $\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \Gamma, \vec{Z} \mathrel{\dot{\notin}} Y, \neg\breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y, \pi^1_{m+2}(Z,s,t)$,

(iii) $\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \Gamma, \vec{Z} \mathrel{\dot{\notin}} X, \neg\breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}X, \neg\pi^1_{m+2}(Z,s,t){\restriction}X$.

By Lemma VI.2.6 (ii) implies (iia) (see below), where we dropped the assumption $\vec{Z}\dot{\in}X$, as this follows form $\vec{Z}\dot{\in}Y$ and $Y\dot{\in}X$ and $\breve{\mathsf{T}}^{+m}_{y(n)}{\restriction}X$. Next, we rewrite (iia) as (iib), and then exploit that if X is a model of $\breve{\mathsf{T}}^{+m}_{y(n)}$, then we have by Lemma VI.2.8 (b) that for all $\vec{Z}\dot{\in}X$ and $\alpha \lhd o(y)$, there is a $Y\dot{\in}X$ with $\vec{Z}\dot{\in}Y$ and $\breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y$. More precisely, $\breve{\mathsf{T}}^{+m}_{y(n)}|\mathsf{S} \vdash \alpha \lhd o(y) \to \exists Y[\vec{Z}\dot{\in}Y \wedge \breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y]$, so by Lemma VI.2.6

$$\mathsf{T}^\epsilon|\mathsf{S} \vdash \alpha \lhd o(y)) \wedge \mathsf{M}_0\dot{\in}X \wedge \breve{\mathsf{T}}^{+m}_{y(n)}{\restriction}X \to \exists Y[\vec{Z}\dot{\in}Y \wedge Y\dot{\in}X \wedge \breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y].$$

This yields (iic).

(iia) $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \mathrel{\dot{\notin}} X, \neg\breve{\mathsf{T}}^{+m}_{y(n)}{\restriction}X, \vec{Z} \mathrel{\dot{\notin}} Y, Y \mathrel{\dot{\notin}} X, \Gamma{\restriction}X, \neg\breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y, \pi^1_{m+2}(Z,s,t){\restriction}X$.

(iib) $\mathsf{T}^\epsilon|\mathsf{S} \vdash \mathsf{M}_0 \mathrel{\dot{\notin}} X, \neg\breve{\mathsf{T}}^{+m}_{y(n)}{\restriction}X, \forall Y\neg[\vec{Z}\dot{\in}Y \wedge Y\dot{\in}X \wedge \breve{\mathsf{T}}^{+m}_{y[\alpha]}{\restriction}Y], \Gamma{\restriction}X, \pi^1_{m+2}(Z,s,t){\restriction}X$.

(iic) $\mathsf{T}^\epsilon|\mathsf{S} \vdash \neg(\alpha \lhd o(y)), \mathsf{M}_0 \mathrel{\dot{\notin}} X, \neg\breve{\mathsf{T}}^{+m}_{y(n)}{\restriction}X, \vec{Z} \mathrel{\dot{\notin}} X, \Gamma{\restriction}X, \pi^1_{m+2}(Z,s,t){\restriction}X$.

Now (iic) and (iii) yield

(iv) $\mathsf{T}^{+m}_{y(n)}|\mathsf{S} \vdash \neg(\alpha \lhd o(y)), \neg B$, for $B := \exists X[\mathsf{M}_0 \dot{\in} X \wedge \vec{Z} \dot{\in} X \wedge \breve{\mathsf{T}}^{+m}_{y[\alpha]} \restriction X \wedge \breve{\mathsf{T}}^{+m}_{y(n)} \restriction X \wedge \neg(\Gamma) \restriction X]$.

Next, we will show that

(v) $\mathsf{T}^{+m}_{y(n+1)} | \mathsf{S} \vdash \neg(\alpha \lhd o(y)), \Gamma, B$.

For that, we work informally in $\mathsf{T}^{+m}_{y(n+1)} | \mathsf{S}$. We assume $\alpha \lhd o(y)$ and $\neg(\Gamma)$. By Lemma VI.2.8 (b), we have $\mathsf{p}_{m+1}(\breve{\mathsf{T}}^{+m}_{y[\alpha]} \wedge \breve{\mathsf{T}}^{+m}_{y(n)})$. Hence, as $\neg(\Gamma)$ is Π^1_{m+1}, B follows. Finally, as $\breve{\mathsf{T}}^{+m}_{y(n+1)}$ implies $\breve{\mathsf{T}}_{y(n)}$, $\mathsf{T}^{+m}_{y(n+1)} | \mathsf{S} \vdash \Gamma$ is by (v), (iv) and (i). □

For the proof of the corollary below, we use these auxiliary properties.

Lemma VI.2.10. *Assume that* $\deg(x) = m{+}2$ *and* $x =_{NF} y \circ_m z$, *and that* $\breve{\mathsf{T}}$ *is* Π^1_2. *Then,*

(c) $\mathsf{T}^{\epsilon} \vdash A_{\breve{\mathsf{T}},x,0}(Z, \alpha, s, t) \leftrightarrow A_{\breve{\mathsf{T}}_z,y,m}(Z, \alpha, s, t)$.

(d) $\mathsf{Op}_{x(\beta)} \Leftrightarrow \mathsf{Op}^{+m}_{y(\beta)} \circ \mathsf{Op}_z$.

Proof (c) As $x =_{NF} y \circ_m z$, $x[\alpha] = y[\alpha] \circ_m z$, so $\deg(x){+}0 = \deg(y){+}m$. Further, by Lemma III.6.10, $\mathsf{Op}_{x[\alpha]}$ iff $\mathsf{Op}^{+m}_{y[\alpha]} \circ \mathsf{Op}_z$. Thus, also $\breve{\mathsf{T}}^{+m}_{x[\alpha]} \restriction X$ iff $\mathsf{Op}^{+m}_{y[\alpha]}(\breve{\mathsf{T}}_z) \restriction X$, and the claim follows. (d) As $x =_{NF} y \circ_m z$, also $x(\beta) = y(\beta) \circ_m z$, and the claim is by Lemma III.6.10. □

Corollary VI.2.11. *Let* $x \in Q$ *with* $\deg(x) = m{+}2$ *and* $x =_{NF} y \circ_m z$. *Further, assume that* $\breve{\mathsf{T}}$ *is* Π^1_2, *and* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$. *Then,*

$$\mathsf{T}_x | \mathsf{S} \mathrel{|\!\!\frac{n}{*}} \Gamma \;\Rightarrow\; \mathsf{T}_{x(n)} | \mathsf{S} \vdash \Gamma.$$

Proof By induction on n. If Γ is an axiom, then as $\Gamma \subseteq \mathcal{L}_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$, Γ is also an axiom of $\mathsf{T}^{+m}_{x(n)} | \mathsf{S}$. Since with $\breve{\mathsf{T}}_{x(n)} | \mathsf{S} \vdash \Gamma$, also $\breve{\mathsf{T}}_{x(n+1)} | \mathsf{S} \vdash \Gamma$, the only interesting case of the induction step is if $\mathsf{T}_x | \mathsf{S} \mathrel{|\!\!\frac{n+1}{*}} \Gamma$ is obtained by a cut with an instances of $\breve{\mathsf{T}}_x | \mathsf{S}$ which is not also an instance of $\breve{\mathsf{T}}_{x(n+1)} | \mathsf{S}$. In this case, $\mathsf{T}_x | \mathsf{S} \mathrel{|\!\!\frac{n}{*}} \Gamma, \neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha)$. As $\neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha) \in \mathcal{L}_{\mathsf{e}\Pi^1_{m+2}}$, also $\mathsf{T}_{x(n)} | \mathsf{S} \mathrel{|\!\!\frac{n}{*}} \Gamma, \neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha)$ by Lemma VI.2.1 and the I.H. Hence also $\mathsf{Op}_{x(n)}(\mathsf{T}) | \mathsf{S} \vdash \Gamma, \neg A_{\breve{\mathsf{T}}_z,y,m}(Z, \alpha)$ by Lemma VI.2.10 (c). As $\mathsf{Op}_{x(n)}(\mathsf{T})$ iff $\mathsf{Op}^{+m}_{y(n)}(\mathsf{T}_z)$ by Lemma VI.2.10 (d), we have $\mathsf{Op}^{+m}_{y(n)}(\mathsf{T}_z) | \mathsf{S} \vdash \Gamma, \neg A_{\breve{\mathsf{T}}_z,y,m}(Z, \alpha)$, hence $\mathsf{Op}^{+m}_{y(n+1)}(\mathsf{T}_z) | \mathsf{S} \vdash \Gamma$ by Lemma VI.2.9. Again by Lemma VI.2.10 (d), we have $\mathsf{Op}^{+m}_{y(n+1)}(\mathsf{T}_z)$ iff $\mathsf{Op}^{+m}_{x(n+1)}(\mathsf{T})$, so $\mathsf{T}_{x(n+1)} | \mathsf{S} \vdash \Gamma$ follows. □

VI.3 Infinitary reductions

We will observe that the results form the previous section hold also (to some extend) in the infinitary systems. In order to fully exploit these results, we require that $\overset{*}{\mathsf{T}}{}^{\epsilon} | \mathsf{S} \mathrel{|\!\!\frac{<\gamma}{*}} \mathsf{M}_0 \models \Gamma$ iff $\overset{*}{\mathsf{S}} \mathrel{|\!\!\frac{<\gamma}{*}} \Gamma$ (Lemma VI.3.13), shown in the next subsection.

VI.3.1 From $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{S} \overset{<\gamma}{\underset{*}{\vdash}} \mathsf{M}_0 \models \Gamma$ to $\overset{*}{\mathsf{S}} \overset{<\gamma}{\underset{*}{\vdash}} \Gamma$

For an arithmetical sequent Γ, we will need the property displayed in the subsection heading to go from $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{<\gamma}{\underset{+}{\vdash}} \Gamma$ via $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{S} \overset{<\gamma'}{\underset{*}{\vdash}} \Gamma$ and $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{S} \overset{<\gamma'}{\underset{*}{\vdash}} \mathsf{M}_0 \models \Gamma$ to $\overset{*}{\mathsf{S}} \overset{<\gamma'}{\underset{*}{\vdash}} \Gamma$. Since the aforementioned and the following results are only required for S different from T^{ϵ} we stick to the following convention.

Convention VI.3.1. *In this subsection, it is assumed that* $\mathsf{S} := \mathsf{T}_{k-1}|\dots|\mathsf{T}_0$ *for* $k > 0$*, so* M_0 *refers to* U_k*.*

We start by refining the definition of $\mathsf{M}_0 \models A$. The purpose is twofold: firstly, we need a definition that is simpler to work with, and secondly, we want that $\mathsf{M}_0 \models A$ stores enough information to easily reconstruct A.

To achieve these goals, we let, $\mathsf{M}_0 \models A$ be an $\mathcal{L}_{\mathsf{M}_0}$-formula which contains $*$-variables $*_i$ that are in some sense substitutes for the set variables U_i, and then identify the $\mathcal{L}_{\mathsf{M}_0}$-formula $\mathsf{M}_0 \models A$ with the $\mathcal{L}$-formula $(\mathsf{M}_0 \models A)^{\Phi_0}$ (see Definition VI.3.3).

$\mathcal{L}_{\mathsf{M}_0}$-formulas are build using a third kind of variables, denoted by $*_0, *_1, \dots$ A variable $*_i$ can be quantified, and it can occur free or bound in a formula in the same may as a number variable x can occur free or bound in an L_2-formula. We let $*$ denote some $*_i$ in the same way we use u to denote some u_i. Further, $\mathsf{FV}_*(\Gamma)$ denotes the set of free $*$-variables that occur in Γ.

Definition VI.3.2 ($\mathcal{L}_{\mathsf{M}_0}$-formulas)**.**

(i) *Each literal of* $\mathcal{L}$ *without set variables, except* $s \in \mathsf{M}_0$ *and* $s \notin \mathsf{M}_0$ *for* $s \in \mathcal{L}$*, is a literal of* $\mathcal{L}_{\mathsf{M}_0}$*.*

(ii) *For each* $s \in \mathcal{L}$*,* $s \in (\mathsf{M}_0)_*$ *and* $s \notin (\mathsf{M}_0)_*$ *are literals of* $\mathcal{L}_{\mathsf{M}_0}$*, where* $s \in (\mathsf{M}_0)_*$ *is short for* $\langle s, * \rangle \in \mathsf{M}_0$*.*

(iii) *The* $\mathcal{L}_{\mathsf{M}_0}$*-formulas are build from the literals by closing under* $\wedge$*,* $\vee$*,* $\mathcal{Q}x$ *and* $\mathcal{Q}*$*, where* $\mathcal{Q} \in \{\forall, \exists\}$*.*

The $\mathcal{L}^*_{\mathsf{M}_0}$*-formulas are the* $\mathcal{L}_{\mathsf{M}_0}$*-formulas without free number variables.*

Using an assignment Φ that maps $*$-variables to closed number terms, we map $\mathcal{L}_{\mathsf{M}_0}$-formulas to $\mathcal{L}$-formulas.

Definition VI.3.3. *Let* Φ *be a map that assigns to each variable* $*_i$ *a closed number term. If* $s \in \mathcal{L}^*$*, then* $\Phi[*_i = s](*_i) = s$*, and if* $i \neq j$*, then* $\Phi[*_i = s](*_j) = \Phi(*_j)$*. Further, each* $\mathcal{L}_{\mathsf{M}_0}$*-formula* A *is mapped to the* $\mathcal{L}$*-formula* A^{Φ} *as follows.*

(i) *For each literal,* L^{Φ} *is obtained form* L *by replacing* $*_i$ *by* $\Phi(*_i)$*.*

(ii) $(A \, j \, B)^\Phi := A^\Phi \, j \, B^\Phi$, $(\mathcal{Q}xA(x))^\Phi := \mathcal{Q}x(A(x))^\Phi$, *and* $(\mathcal{Q}{*}A(*))^\Phi := \mathcal{Q}x(A(x))^\Phi$, *where x is the first number variable w.r.t. some fixed enumeration that does not occur in* $(A(0))^\Phi$.

If the context suggest that the $\mathcal{L}_{\mathsf{M}_0}$-formula A should be read as an $\mathcal{L}$-formula, then we identify A with A^{Φ_0} where $\Phi_0(*_n) := \overline{n}$, and if the context further suggest that A is an $\mathcal{L}^*$-formula, then we identify A with $(A^{\Phi_0})^*$.

Next, we assign to each $\mathcal{L}$-formula A an $\mathcal{L}_{\mathsf{M}_0}$-formula $\mathsf{M}_0 \models A$ with the same free number variables, and conversely, we assign to each $\mathcal{L}_{\mathsf{M}_0}$-formula B an $\mathcal{L}$-formula $B^\uparrow$ with the same free number variables. It is readily observed that $(\mathsf{M}_0 \models A)^\uparrow = A$ and $B = (\mathsf{M}_0 \models B^\uparrow)$.

Definition VI.3.4 ($\mathsf{M}_0 \models A$)**.** *To each $\mathcal{L}$-formula A, we assign an $\mathcal{L}_{\mathsf{M}_0}$-formula $\mathsf{M}_0 \models A$ as follows.*

(i) $\mathsf{M}_0 \models L := L$ *if L is a literal (without a set variable).*

(ii) $\mathsf{M}_0 \models (s \in U_i) := s \in (\mathsf{M}_0)_{*_i}$ *and* $\mathsf{M}_0 \models (s \notin U_i) := s \notin (\mathsf{M}_0)_{*_i}$.

(iii) $\mathsf{M}_0 \models \mathcal{Q}xA(x) := \mathcal{Q}x(\mathsf{M}_0 \models A(x))$ *and*
$\mathsf{M}_0 \models (A \, j \, B) := (\mathsf{M}_0 \models A) \, j (\mathsf{M}_0 \models B)$, *where* $j \in \{\wedge, \vee\}$.

(iv) $\mathsf{M}_0 \models \mathcal{Q}U_iA(U_i) := \mathcal{Q}{*_i}(\mathsf{M}_0 \models A(U_i))$.

Definition VI.3.5 ($A^\uparrow$)**.** *To each $\mathcal{L}_{\mathsf{M}_0}$-formula A, we assign an $\mathcal{L}$-formula $A^\uparrow$ as follows.*

(i) $L^\uparrow := L$, *if L is a literal without a $*$-variable.*

(ii) $(s \in (\mathsf{M}_0)_{*_i})^\uparrow := s \in U_i$ *and* $(s \notin (\mathsf{M}_0)_{*_i})^\uparrow := s \notin U_i$.

(iii) $(\mathcal{Q}xA)^\uparrow := \mathcal{Q}xA^\uparrow$ *and* $(A \, j \, B)^\uparrow := A^\uparrow \, j \, B^\uparrow$, *where* $j \in \{\wedge, \vee\}$.

(iv) $(\mathcal{Q}{*_i}A)^\uparrow := \mathcal{Q}U_iA^\uparrow$.

Now the following is readily checked.

Lemma VI.3.6. $\mathsf{M}_0 \models \Gamma$ *according to the old definition (Definition I.1.5) agrees with $\mathsf{M}_0 \models \Gamma$ according to the new definition (Definition VI.3.4) (up to names of bound variables). Note that here $\mathsf{M}_0 \models \Gamma$ is short for $(\mathsf{M}_0 \models \Gamma)^{\Phi_0}$.*

If A is an $\mathcal{L}(\mathsf{S})$-formula without dummy set quantifiers (i.e. A contains no subformula of the form $\mathcal{Q}XA(X)$ where X does not occur free in $A(X)$), then we call $\mathsf{M}_0 \models A$ an $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formula. It is easily seen that $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formulas can also be characterized as follows.

Definition VI.3.7. *An $\mathcal{L}_{\mathsf{M}_0}$-formula A is an $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formula, if $A^{\uparrow} \in \mathcal{L}(\mathsf{S})$ and if A contains no subformula of the form $\mathcal{Q}{*}A(*)$ where $*$ does not occur free in $A(*)$ (no dummy $*$-quantifiers).*

A key property of $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formulas is that $A^{\Phi_0} = B^{\Phi_0}$ entails $A = B$ (we removed formulas with dummy $*$-quantifier from the $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formulas, as $(\forall *_0(0=0))^{\Phi_0}$ is $(\forall u_0(0=0))^{\Phi_0}$). To show this, we need the following auxiliary lemma.

Lemma VI.3.8. *Assume that $u \in \mathsf{FV}_0(A(u))$. If $A(*) \in \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, then $A(t) \notin \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, and if $A(t) \in \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, then $A(*) \notin \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$.*

Proof By induction on the build-up of A. The claim is readily checked for literals. Note that $(s \in (\mathsf{M}_0)_*) \in \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, but $(s \in (\mathsf{M}_0)_t) \notin \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, and conversely, for $i < k$ where U_i is different from M_0, $s \in (\mathsf{U}_i)_t \in \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$, but $s \in (\mathsf{U}_i)_* \notin \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$. The induction step is straightforward. □

Lemma VI.3.9. *If $A, B \in \mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$ and $A^{\Phi_0} = B^{\Phi_0}$, then $A = B$ (up to names of bound $*$-variables).*

Proof By induction on the build-up of $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$-formulas. If $L = A^{\Phi_0}$ is a literal, then A can only differ from B if L is of the form $L(\overline{m})$. But by Lemma VI.3.8 only either $L(\overline{m})$ or $L(*_m)$ is literal of $\mathcal{L}_{\mathsf{M}_0}(\mathsf{S})$. If e.g. $A^{\Phi_0} = B^{\Phi_0}$ and $A = \mathcal{Q}{*}C(*)$, then $B = \mathcal{Q}xD(x)$ is impossible: Since $\mathcal{Q}*$ is not a dummy quantifier, $*$ occurs free in $C(*)$. And as $\cdot^{\Phi_0}$ does not change the structure of the formula, also x occurs free in $D(x)$. Further, $A^{\Phi_0} = B^{\Phi_0}$ implies $(C(*))^{\Phi_0} = (D(x))^{\Phi_0}$, so by I.H., $C(*) = D(x)$, a contradiction! The other cases are shown similarly. □

Next, we modify $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S}$ so that it derives $\mathcal{L}^*_{\mathsf{M}_0}$-sequents. We call this system $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S}$.

Definition VI.3.10. *$\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S}$ derives finite set of $\mathcal{L}^*_{\mathsf{M}_0}$-formulas. It contains the axioms and rules of $\overset{*}{\mathsf{T}}{}^{\epsilon}$ adjusted to $\mathcal{L}_{\mathsf{M}_0}$-sequents, for each $A \in \mathsf{inst}^*(\breve{\mathsf{S}})$ an axiom $\Gamma, \mathsf{M}_0 \models A$, and the $\mathcal{Q}*$-rules*

$$\frac{\Gamma, A(*_i)}{\Gamma, \forall *_i, A(*_i)}, \quad \text{and} \quad \frac{\Gamma, A(*_i)}{\Gamma, \exists *_i, A(*_i)},$$

where $*_i$ does not occur in $\Gamma, \forall *_i A(*_i)$.

Note that $\mathsf{inst}^*(\breve{\mathsf{S}}{\restriction}\mathsf{M}_0) = \{(\mathsf{M}_0 \models A)^{\Phi_0} : A \in \mathsf{inst}^*(\breve{\mathsf{S}})\}$. Therefore, $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S}$ is essentially $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S}$, with the $\mathcal{Q}X$-rules replaced by the $\mathcal{Q}*$-rules.

The following is immediate by induction the depth of the derivation.

Lemma VI.3.11. *Let $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*(\mathsf{S})$. Then,*

$$\overset{*}{\mathsf{S}} \,\frac{\alpha}{*}\, \Gamma \Leftrightarrow \overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S} \,\frac{\alpha}{*}\, \mathsf{M}_0 \models \Gamma,$$

However, we want to show that for a finite set Γ of $\mathcal{L}^*_{\mathsf{M}_0}(\mathsf{S})$-formulas, $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma^{\Phi_0}$ implies $\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma^{\uparrow}$. This follows rather straightforward since for $A \in \mathcal{L}^*_{\mathsf{M}_0}(\mathsf{S})$, A^{Φ_0} already determines A due to Lemma VI.3.9.

Lemma VI.3.12. *Let $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{M}_0}(\mathsf{S})$. Then, all of following statements are equivalent.*

(i) $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma$,

(ii) $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma^{\Phi}$ *for each* Φ,

(iii) $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma^{\Phi_0}$.

Proof (i)$\Rightarrow$(ii) is immediate by induction on α, and (ii)$\Rightarrow$(iii) holds trivially. It remains to show (iii)$\Rightarrow$(i). We prove the claim by induction on α. If $\alpha = 0$, then Γ^{Φ_0} contains $L^{\Phi_0}, \neg L^{\Phi_0}$ or an $\mathcal{L}^*$-instance of $\breve{\mathsf{S}}\restriction\mathsf{M}_0$, which is $(\mathsf{M}_0 \models A)^{\Phi_0}$ for some $A \in \mathsf{inst}^*(\breve{\mathsf{S}})$. By Lemma VI.3.9, Γ is also an axiom of $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S}$. Of the induction step, we exemplarily consider the case where $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha+1}{*}} (\Gamma, \forall x A(x))^{\Phi_0}$ is obtained by an ω-rule from $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} (\Gamma, A(\overline{m}))^{\Phi_0}$ for all $m \in \mathbb{N}$. Then, $(\forall x A(x))^{\Phi_0}$ is either of the form $(\forall * A(*))^{\Phi_0}$ or $(\forall x A(x))^{\Phi_0}$, and due to Lemma VI.3.9, only one is possible. In the first case, the I.H. yields $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma, A(*)$ for some $*$ that does not occur free in $\Gamma, \forall * A(*)$, and the claim follows by an $\forall *$-rule. And in the second case, the I.H. yields $\overset{*}{\mathsf{T}}{}^{\epsilon\downarrow}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma, A(\overline{m})$ for each $m \in \mathbb{N}$, and the claim follows by an application of the ω-rule. $\square$

Combining the two above lemmas into one yields the following.

Lemma VI.3.13. *Let $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*(\mathsf{S})$ without dummy set quantifiers. Then,*

$$\overset{*}{\mathsf{S}} \mathrel{|\!\!\frac{\alpha}{*}} \Gamma \;\Leftrightarrow\; \overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{|\!\!\frac{\alpha}{*}} \mathsf{M}_0 \models \Gamma.$$

VI.3.2 Reducing $\overset{*}{\mathsf{T}}_{x+1}$ to $\overset{*}{\mathsf{T}}_{x}$ and if $\deg(x) > 1$, then $\overset{*}{\mathsf{T}}_{x}$ to $\overset{*}{\mathsf{T}}_{x(\alpha)}$

We present infinitary versions of the Lemmas VI.2.3, VI.2.9 and Corollary VI.2.11. These results are obtained very similarly to the finitary versions. The following variant of Lemma VI.2.1 is used which is proved completely analogously.

Lemma VI.3.14. *If for some α, β and each $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_m}$, $\overset{*}{\mathsf{S}} \mathrel{|\!\!\frac{\alpha}{\mathbb{C}}} \Gamma \Rightarrow \overset{*}{\mathsf{S}}{}' \mathrel{|\!\!\frac{\beta}{\mathbb{C}}} \Gamma$, then also for each $\Delta \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Pi^1_{m+1}}$, $\overset{*}{\mathsf{S}} \mathrel{|\!\!\frac{\alpha}{\mathbb{C}}} \Delta \Rightarrow \overset{*}{\mathsf{S}}{}' \mathrel{|\!\!\frac{\beta}{\mathbb{C}}} \Delta$.*

Further, we need an infinitary versions of Lemmas V.2.8 (iii). Here, we need just the case $n = 0$. The proof is easily adapted.

Lemma VI.3.15. *If* $\Gamma \subseteq \mathcal{L}^*(\mathsf{T}^2|\mathsf{S})$*, then*

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}|\mathsf{S} \,\frac{\alpha}{\mathbb{C}}\, \Gamma \Longrightarrow \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, \Gamma[\mathsf{M}_2/\mathsf{M}_1].$$

In the proof of the next lemma, we cut with formulas of the form $(\mathsf{M}_0)_s = (\mathsf{M}_0)_s$ and $Z\dot{\in}\mathsf{M}_0$, and further, with the formula B specified at $(*)$ in the proof below. Therefore, we assume that $\mathbb{C}$ in the lemma below is so that $\mathsf{cut}(\mathbb{C}, \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^n|\mathsf{S})$ contains these formulas. There will be enough room to eliminate these cuts in a further step. Also note that now $\mathsf{M}_0 \models \Gamma$ is an $\mathcal{L}^*_{\mathsf{M}_0}$-sequent which we identify with the $\mathcal{L}^*$-sequent $(\mathsf{M}_0 \models \Gamma)^{\Phi_0}$.

Lemma VI.3.16. *Let* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$ *and* $\mathbb{C} \subseteq \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$ *so that* $\mathsf{cut}(\mathbb{C}, \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}|\mathsf{S})$ *contains all formulas of the form* $(\mathsf{M}_0)_s = (\mathsf{M}_0)_s$ *and* $Z\dot{\in}\mathsf{M}_0$*, and the formula* B *specified at* $(*)$ *in the proof below. Then,*

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}|\mathsf{S} \,\frac{\alpha}{\mathbb{C}}\, \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \mathsf{M}_1 \models \Gamma \;\Rightarrow\; \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, \mathsf{M}_1 \models \Gamma.$$

Proof Assume that $\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^n|\mathsf{S} \,\frac{\alpha}{\mathbb{C}}\, \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z), \mathsf{M}_1 \models \Gamma$, where we suppose that $\mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}})}(Z)$ is $\forall x[(\mathsf{M}_1)_* \dot{\notin} (\mathsf{M}_1)_x \vee \neg\breve{\mathsf{T}}\restriction(\mathsf{M}_1)_x]$ and $*' \notin \mathsf{FV}_*(\mathsf{M}_1 \models \Gamma)$. By Lemma VI.3.11, we also have for each Φ,

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{\alpha}{\mathbb{C}}\, ((\mathsf{M}_1)_* \dot{\notin} (\mathsf{M}_1)_{*'} \vee \neg\breve{\mathsf{T}}\restriction(\mathsf{M}_1)_{*'})^\Phi, (\mathsf{M}_1 \models \Gamma)^\Phi.$$

By Lemma V.3.12 we obtain that for all $s, t \in \mathcal{L}^*$,

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, Z \neq (\mathsf{M}_1)_s, X \neq (\mathsf{M}_1)_t, Z \dot{\notin} X, \neg\breve{\mathsf{T}}\restriction X, (\mathsf{M}_1 \models \Gamma)[Z/(\mathsf{M}_1)_*],$$

which readily yields

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, Z \dot{\notin} \mathsf{M}_1, X \dot{\notin} \mathsf{M}_1, Z \dot{\notin} X, \neg\breve{\mathsf{T}}\restriction X, (\mathsf{M}_1 \models \Gamma)[Z/(\mathsf{M}_1)_*].$$

Thus, since $\mathsf{T}^\epsilon|\mathsf{T}|\mathsf{S} \vdash X\dot{\in}\mathsf{M}_1 \wedge Z\dot{\in}X \to Z\dot{\in}\mathsf{M}_1$, and since $Z\dot{\in}\mathsf{M}_1 \in \mathsf{cut}(\mathbb{C}, \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}|\mathsf{S})$,

$$\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, X \dot{\notin} \mathsf{M}_1, Z \dot{\notin} X, \neg\breve{\mathsf{T}}\restriction X, (\mathsf{M}_1 \models \Gamma)[Z/(\mathsf{M}_1)_*].$$

Using Lemma VI.3.15, we obtain

$$(*) \qquad \overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, \forall X(Z\dot{\in}X \wedge X\dot{\in}\mathsf{M}_2 \wedge \breve{\mathsf{T}}\restriction X \to (\mathsf{M}_2 \models \Gamma)[Z/(\mathsf{M}_1)_*]) =: B.$$

As we have $\mathsf{T}^\epsilon|\mathsf{T}^2|\mathsf{S} \vdash \neg B, \mathsf{M}_1 \models \Gamma$ (by the proof of Lemma VI.2.3), and thus $\mathsf{T}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\omega}{*}\, \neg B, \mathsf{M}_1 \models \Gamma$, Lemma V.3.15 yields $\overset{*}{\dot{\mathsf{T}}}{}^\epsilon|\mathsf{T}^2|\mathsf{S} \,\frac{<\alpha^+}{\mathbb{C}}\, \mathsf{M}_1 \models \Gamma$. $\square$

For the infinitary variant of Lemma VI.2.9, we use the following infinitary variant of Lemma VI.2.6.

Lemma VI.3.17. *If* $\overset{*}{\mathsf{T}}|\mathsf{S} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma$ *and* $Y \notin \mathsf{FV}_1(\Gamma) = \{Z_1, \ldots, Z_k\}$ *and* $\mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S}) \subseteq \mathbb{C}$, *then*

$$\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \mathsf{M}_0 \,\dot{\notin}\, Y, \vec{Z} \notin Y, \neg\breve{\mathsf{T}}{\restriction}Y, \Gamma{\restriction}Y.$$

Proof If $\overset{*}{\mathsf{T}}|\mathsf{S} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma$, then by Lemma V.3.9, $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \neg\breve{\mathsf{T}}, \forall X[X \neq \mathsf{M}_0], \Gamma$. Now Lemma V.3.11 yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \neg\breve{\mathsf{T}}{\restriction}Y, \forall X[X \neq \mathsf{M}_0]{\restriction}Y, \vec{Z} \,\dot{\notin}\, Y, \Gamma{\restriction}Y$. Since further, $\forall X[X \neq \mathsf{M}_0]{\restriction}Y$ is $\mathsf{M}_0 \notin Y$, the claim follows. □

Lemma VI.3.18. *Let* y *be a simple name of degree two. Further, assume that* $\breve{\mathsf{T}}$ *is* Π^1_{m+2} *and* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$ *and* $\mathbb{C} \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$. *Then, for each* $\alpha < o(y)$,

$$\overset{*}{\mathsf{T}}{}^{+m}_{y(\beta)}|\mathsf{S} \,|\!\frac{\alpha}{\mathbb{C}}\, \Gamma, \neg A_{\breve{\mathsf{T}},y,m}(Z, \alpha, s, t) \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^{+m}_{y(\beta+1)}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \Gamma.$$

Proof Almost literally as in the finitary case. We just use Lemma VI.3.17 instead of Lemma VI.2.6. Further, we use that if $\mathsf{T}|\mathsf{S} \vdash \Gamma, A$, then also $\overset{*}{\mathsf{T}}|\mathsf{S} \,\Big|\!\frac{<\omega}{*}\, \Gamma, A$, and if further $\overset{*}{\mathsf{T}}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \Gamma, \neg A$ and $\mathsf{sufo}^-(A) \subseteq \mathbb{C}$, then by Corollary V.4.4, also $\overset{*}{\mathsf{T}}|\mathsf{S} \,\Big|\!\frac{<\alpha^+}{\mathbb{C}}\, \Gamma$ (actually, when this situation applies, A is a Σ^1_1-formula, so $A^\circ = A$ and $\mathsf{sufo}^-(A)$ are arithmetical). □

Corollary VI.3.19. *Let* $x \in Q$ *with* $\deg(x) = m{+}2$. *Further, assume that* $\breve{\mathsf{T}}$ *is* Π^1_2, *and* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$. *Then,*

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\Big|\!\frac{\beta}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}}\, \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,\Big|\!\frac{<\beta^+}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}}\, \Gamma.$$

Proof By Lemma VI.1.3 it suffice to show the claim for $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S}$, where still, $\breve{\mathsf{T}}'_x := (\forall\alpha < o(x))\mathsf{p}_{m+2}(\overset{*}{\mathsf{T}}_{x[\alpha]})$. We let $\mathbb{C} := \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}$. Recall that then $\mathsf{inst}^*(\overset{*}{\mathsf{T}}{}'_{x(\beta)}|\mathsf{S}) \subseteq_{\mathsf{fin}} \mathbb{C}$ (cf. Lemma VI.1.2). The claim is now shown by induction on β, very similar as in the finitary case, using Lemma VI.3.14 in place of Lemma VI.2.1. We just hint at some differences.

If β is a limit, then by Lemma V.3.6, we may assume that $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,|\!\frac{\beta}{\mathbb{C}}\, \Gamma, \forall x B(x)$ is obtained from $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\Big|\!\frac{<\beta}{\mathbb{C}}\, \Gamma, B(\overline{m})$, for all $m \in \mathbb{N}$. Since $\overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,\Big|\!\frac{<\omega}{*}\, \breve{\mathsf{T}}_{x(\alpha)}|\mathsf{S}$ for each $\alpha < \beta$, we have by Lemma V.4.6, that for each Δ and γ, $\overset{*}{\mathsf{T}}_{x(\alpha)}|\mathsf{S} \,\Big|\!\frac{<\gamma}{\mathbb{C}}\, \Delta \Rightarrow \overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,\Big|\!\frac{<\gamma}{\mathbb{C}}\, \Delta$. Using the I.H. we obtain that $\overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,\Big|\!\frac{<\beta}{\mathbb{C}}\, \Gamma, B(\overline{m})$, for all $m \in \mathbb{N}$, hence also $\overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,|\!\frac{\beta}{\mathbb{C}}\, \Gamma, \forall x B(x)$.

Further, we need that $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\Big|\!\frac{<\beta^+}{\mathbb{C}}\, \Gamma, \neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha)$ yields $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\Big|\!\frac{<\beta^+}{\mathbb{C}}\, \Gamma, \neg A_{\breve{\mathsf{T}}_z,y,m}(Z, \alpha)$. As $\neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha) \in \mathcal{L}_{\mathsf{e}\Pi^1_{m+2}}$ this is either directly by Lemma VI.2.10 (c) and Lemma V.4.3, or alternatively, one may observe that $\neg A_{\breve{\mathsf{T}},x,0}(Z, \alpha)$ and $\neg A_{\breve{\mathsf{T}}_z,y,m}(Z, \alpha)$ only differ in the subformulas $\neg(\breve{\mathsf{T}})^{+0}_{x[\alpha]}\restriction X$ and $\neg(\breve{\mathsf{T}}_z)^{+m}_{y[\alpha]}\restriction X$, which are equivalent over $\mathsf{T}_{x(\beta)}|\mathsf{S}$ for $\alpha \lhd o(y)$. Hence we can employ inversion to dig out $\neg(\breve{\mathsf{T}})^{+0}_{x[\alpha]}\restriction X$, replace it by $\neg(\breve{\mathsf{T}}_z)^{+m}_{y[\alpha]}\restriction X$ by cutting, and then undo the inversion steps. $\square$

VI.4 Bounds

Finally, we prove that for each $x \in Q$, g_{x^h} is a sharp bound of T_x, so according to our provisional definition of sharp bounds, g_{x^h} is the largest normal function which is provable in T_x, or more precisely, for any other normal function f that is provable in T_x, we have that $f\restriction\mathsf{Lim}(\Omega) \leq g_{x^h}\restriction\mathsf{Lim}(\Omega)$, where now again, $g(\alpha) := \omega^{1+\alpha}$ and $\mathsf{T} := \mathsf{ACA}_0$ is fixed for this and the final section.

We prove this result by showing that g_{x^h} is a bound of T_x. A bound of T_x ceils the costs of cut-elimination for derivations $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{<\gamma}{\underset{+}{\vdash}} \Gamma$, where the cut-rule is restricted to formulas in $\mathsf{inst}^*(\breve{\mathsf{T}}_x|\mathsf{S})$ and some additional formulas that do not impede the cut-elimination process; these additional cuts can be eliminated cheaply at a later stage. The next two definitions explain the concept of a bound.

Definition VI.4.1 ($\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\alpha}{\underset{+}{\vdash}} \Gamma$)**.** *Let $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*$ and $x \in Q$.*

(i) If $\deg(x) = m{+}1$, then $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\alpha}{\underset{+}{\vdash}} \Gamma$ states that $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\alpha}{\underset{\mathbb{C}}{\vdash}} \Gamma$ for $\mathbb{C} := \mathcal{L}^*_{\mathsf{e}\Sigma^1_m}$.

(ii) $\mathsf{T}_{q_0}|\mathsf{S} \overset{\alpha}{\underset{+}{\vdash}} \Gamma$ states $\mathsf{T}_{q_0}|\mathsf{S} \overset{\alpha}{\underset{*}{\vdash}} \Gamma$.

For the next definition, we let $m \dot{-} 1 := m{-}1$ if $m > 0$, and $0 \dot{-} 1 := 0$.

Definition VI.4.2 (Bound of T_x)**.** *Assume that $x \in Q$ with $\deg(x) = m$. A normal function $f : \Omega \to \Omega$ is a bound of T_x, if for each S, each $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m\dot{-}1}}(\mathsf{T}_x|\mathsf{S})$ and each $\gamma \in \mathsf{Lim}(\Omega)$,*

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{<\gamma}{\underset{+}{\vdash}} \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \overset{<f(\gamma)}{\underset{*}{\vdash}} \Gamma.$$

f is a weak bound if the above only holds for $\mathsf{S} = \mathsf{T}^\epsilon$, (i.e. $\overset{*}{\mathsf{T}}_x \overset{<\gamma}{\underset{+}{\vdash}} \Gamma \Rightarrow \overset{*}{\mathsf{T}}^\epsilon \overset{<f(\gamma)}{\underset{*}{\vdash}} \Gamma$). f is a sharp bound, if $f\restriction\mathsf{Lim}(\Omega) \leq h\restriction\mathsf{Lim}(\Omega)$ for some h that is provable in T_x.

Remark VI.4.3. *If for each Γ, $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{<\gamma}{\underset{+}{\vdash}} \Gamma \Rightarrow \mathsf{T}^\epsilon|\overset{*}{\mathsf{S}} \overset{<f(\gamma)}{\underset{*}{\vdash}} \Gamma$, then also for each Γ, $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\gamma}{\underset{+}{\vdash}} \Gamma \Rightarrow \mathsf{T}^\epsilon|\overset{*}{\mathsf{S}} \overset{f(\gamma)}{\underset{*}{\vdash}} \Gamma$, by Lemma V.3.6 and as f is continuous.*

A key property of bounds is that a function that is provable in T_x is majorized already by each weak bound of T_x. This is, as we will see, essentially a consequence of Pohlers' Boundedness Lemma (cf. e.g. [11]). Below, we give a slightly strengthened version which is due to Beckmann [1]. Recall that $\mathsf{TI}_\lhd(\mathsf{U},\beta)$ is $\mathsf{Prog}_\lhd(\mathsf{U}) \to \beta \in \mathsf{U}$.

Lemma VI.4.4 (Boundedness Lemma)**.** *If* $\overset{*}{\mathsf{T}}^{\epsilon} \frac{\alpha}{*} \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \beta \in \mathsf{U}$*, then* $\beta \leq \alpha$*.*

For the aforementioned majorization property, we need the following observations.

Lemma VI.4.5. *If* $\deg(x) \leq 1$*, then we let* $\mathcal{C}'_x := \{\alpha : \mathsf{Wo}_{\lhd}(g_{x^h}(\alpha))\}$*, and if* $\deg(x) > 1$*, then* $\mathcal{C}'_x := \{\alpha : \breve{\mathsf{T}}_{x(\alpha)}\}$*.*

(i) $\mathsf{T}_x \vdash \mathcal{C}'_x = \mathcal{C}_x$ *and* $\mathsf{T}_x \vdash \alpha \in \mathcal{C}'_x \to \mathsf{Wo}_{\lhd}(\alpha)$*.*

(ii) If $\deg(x) \leq 1$*, then* $(\alpha \in \mathcal{C}'_x) \in \mathcal{L}^*_{\Pi^1_1}$ *and* $\mathsf{sufo}^-(\alpha \notin \mathcal{C}'_x)^\circ \subseteq \mathcal{L}^*_{\Pi^1_0}$*, and*
if $\deg(x) = m{+}2$*, then* $(\alpha \in \mathcal{C}'_x) \in \mathcal{L}^*_{\mathsf{e}\Pi^1_{m+2}}$ *and* $\mathsf{sufo}^-(\alpha \notin \mathcal{C}'_x)^\circ \subseteq \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}$*.*

(iii) $\overset{*}{\mathsf{T}}_x \frac{<\omega}{+} \alpha \in \mathcal{C}'_x$ *and* $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \alpha \notin \mathcal{C}'_x, \Gamma$ *imply* $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \Gamma$*.*

(iv) If $\alpha < \gamma$*, then* $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \alpha \in \mathcal{C}'_x$*.*

Proof (i) The first claim is trivial, see Definition III.7.2. For the second claim, note that if $\deg(x) \leq 1$, then $\alpha \in \mathcal{C}'_x$ says $\mathsf{Wo}_{\lhd}(g_{x^h}(\alpha))$, thus also $\mathsf{Wo}_{\lhd}(\alpha)$, and if $\deg(x) > 1$, then $\alpha \in \mathcal{C}'_x$ says $\breve{\mathsf{T}}_{x(\alpha)}$, which by its definition implies $\mathsf{Wo}_{\rightsquigarrow^*}(x(\alpha))$, which in turn yields $\mathsf{Wo}_{\lhd}(\alpha)$, as for each $\beta \lhd \alpha$, $x(\beta) \rightsquigarrow^* x(\alpha)$ (cf. Lemma III.4.19 (xi)). (ii) by inspection of the definition of $\mathcal{C}'_x$. (iii) By (ii) and Lemma V.4.3. (iv) By induction on $\alpha < \gamma$. Assume that $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \beta \in \mathcal{C}'_x$ for each $\beta < \alpha$. Then also $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} A := (\forall \beta \lhd \alpha)(\beta \in \mathcal{C}'_x)$. Since $\mathsf{T}_x \vdash \mathsf{Prog}_{\lhd}(\mathcal{C}'_x)$, also $\overset{*}{\mathsf{T}}_x \frac{<\omega}{+} \neg A, \alpha \in \mathcal{C}'_x$. Note that if $(\beta \in \mathcal{C}'_x) \in \mathcal{L}^*_{\mathsf{e}\Pi^1_{m+2}}$, then also $A \in \mathcal{L}^*_{\mathsf{e}\Pi^1_{m+2}}$ and $\mathsf{sufo}^-(A^\circ) \subseteq \mathcal{L}^*_{\mathsf{e}\Pi^1_{m+1}}$. Hence, Lemma V.4.3 yields $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \alpha \in \mathcal{C}'_x$. □

Theorem VI.4.6. *If* $x \in Q$ *and* f *a weak bound of* T_x*, then* $g_{x^h} \restriction \mathsf{Lim}(\Omega) \leq f \restriction \mathsf{Lim}(\Omega)$*.*

Proof Assume that f is a weak bound of T_x. By Corollary III.7.16, T_x proves g_{x^h}, that is, $\mathsf{T}_x \vdash \mathsf{Wo}_{\lhd}(\alpha) \wedge \mathsf{TI}_{\lhd}(\mathcal{C}'_x, \alpha) \to \mathsf{Wo}_{\lhd}(g_{x^h}(\alpha))$. Noting that $\neg\mathsf{TI}_{\lhd}(\mathcal{C}'_x, \alpha)$ is $\mathsf{Prog}_{\lhd}(\mathcal{C}'_x) \wedge \alpha \notin \mathcal{C}'_x$, we also have $\mathsf{T}_x \vdash \neg\mathsf{Wo}_{\lhd}(\alpha), \alpha \notin \mathcal{C}'_x, \mathsf{TI}_{\lhd}(\mathsf{U}, g_{x^h}(\alpha))$. As $\mathsf{T}_x \vdash \alpha \notin \mathcal{C}'_x, \mathsf{Wo}_{\lhd}(\alpha)$ by Lemma VI.4.5 (i), $\mathsf{T}_x \vdash \alpha \notin \mathcal{C}'_x, \mathsf{TI}_{\lhd}(\mathsf{U}, g_{x^h}(\alpha))$ follows.
Now let $\alpha < \gamma$ and $\lambda := g_{x^h}(\alpha)$. By the above, $\overset{*}{\mathsf{T}}_x \frac{<\omega}{*} \alpha \notin \mathcal{C}'_x, \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \lambda \in \mathsf{U}$, and $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \alpha \in \mathcal{C}'_x$ by Lemma VI.4.5 (iv). Now Lemma VI.4.5 (iii) yields $\overset{*}{\mathsf{T}}_x \frac{<\gamma}{+} \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \lambda \in \mathsf{U}$. As f is a bound of T_x, we have $\mathsf{T}^{\epsilon} \frac{<f(\gamma)}{*} \neg\mathsf{Prog}_{\lhd}(\mathsf{U}), \lambda \in \mathsf{U}$. By the Boundedness Lemma, $g_{x^h}(\alpha) < f(\gamma)$, and since g_{x^h} is normal, $g_{x^h}(\gamma) \leq f(\gamma)$. □

Next, we fix an axiomatizations of T_{q_0}, that is, of ACA_0, and then compute a bound of ACA_0. We find it convenient to work with the following finite axiomatization.

Definition VI.4.7. *Let* $\breve{\mathsf{T}}_{q_0} := (\mathsf{ACA}) := \forall X \mathsf{IND}(X) \wedge A_1 \wedge A_2 \wedge A_3 \wedge A_4$*, where* $\mathsf{IND}(U) := \forall x[0 \in U \wedge \forall y(y \in U \to y{+}1 \in U) \to x \in U]$*, and the instances of* A_i *(*$1 \leq i \leq 4$*) have the forms* $\exists X[X = \mathcal{C}_i]$*, where* $\mathcal{C}_1 = \{0\}$*,* $\mathcal{C}_2 = \{x : x \notin Y\}$*, and*

$$\mathcal{C}_3 = \{x : x \in (Y)_0 \vee x \in (Y)_1\} \text{ and } \mathcal{C}_4 = \{x : \{e\}(x, y) \in Y\},$$

where $\{e\}(\langle \vec{x} \rangle, \langle y \rangle) \in Y$ *is a* Σ^0_1*-formula which states that there is a computation of the recursive function with index* e *that yields on input* (x, y) *a value in the set* Y*.*

The idea is simple: A_2 tells us the $\{x : x \notin U\}$ is a set. A_1 and A_4 imply in particular, that for each atom $R(\vec{u})$, $Z := \{\langle \vec{x} \rangle : R(\vec{x})\} = \{\langle \vec{x} \rangle : \mathsf{ch}_R(\langle \vec{x} \rangle) = 0\}$ is a set. Then, also by A_4, $\{y : \langle \vec{x}, y, \vec{z} \rangle \in Z\} = \{y : R(\vec{x}, y, \vec{z})\}$ is a set, too. And using A_3, we can code two sets into one. This allows to prove by induction on the build-up of $A(\vec{V}, \vec{v}, u)$ that for all $\vec{Y}, \vec{y}$, there exists a set $\{x : A(\vec{Y}, \vec{y}, x)\}$.

Next, we compute a bound of T_{q_0}, using the axiomatization of Definition VI.4.7. Note however, that for any alternative axiomatization $\breve{\mathsf{T}}'_{q_0}$ with $\mathsf{inst}^*(\breve{\mathsf{T}}'_{q_0}) \subseteq \mathcal{L}^*_{\Sigma^1_1}$, cut-elimination comes at the same costs due to Lemma V.4.6: if $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*(\mathsf{T}|\mathsf{S})$ and $\overset{*}{\mathsf{T}}{}'_{q_0}|\mathsf{S} \mathrel{\substack{<\gamma \\ \mid\!\overline{} \\ \mathbb{C}}} \Gamma$ for some $\mathbb{C} \subseteq \mathcal{L}^*_{\Pi^1_0}$ with $\mathsf{rk}(\mathbb{C}) < N_0$, then also $\overset{*}{\mathsf{T}}_{q_0}|\mathsf{S} \mathrel{\substack{<\gamma \\ \mid\!\overline{} \\ \mathbb{D}}} \Gamma$ for some $\mathbb{D} \subseteq \mathcal{L}^*_{\Pi^1_0}$ with $\mathsf{rk}(\mathbb{D}) < N'_0$.

Since $A \in \mathsf{inst}^*(\mathsf{S}{\restriction}\mathsf{M}_0) \cup \mathsf{cut}(\mathbb{C}, S)$ contains no free set variables, the following auxiliary claim is immediate by induction on α.

Lemma VI.4.8. *Suppose that* $\overset{*}{\mathsf{T}}{}^{\epsilon} \mathrel{\substack{N_0 \\ \mid\!\overline{} \\ -}} s \in \mathcal{C}, s \notin \mathcal{C}$*. Then, for each arithmetical* Γ*,*

$$\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{\substack{\alpha \\ \mid\!\overline{} \\ *}} \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{\substack{N_0+\alpha \\ \mid\!\overline{} \\ *}} \Gamma[\mathcal{C}/U].$$

Lemma VI.4.9. *Assume that* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$ *and* $\mathbb{D} \subseteq \mathcal{L}^*_{\Pi^1_0}$ *Further, let* N_0 *so that for each* $i \in I$*,* $\overset{*}{\mathsf{T}}{}^{\epsilon} \mathrel{\substack{<N_0 \\ \mid\!\overline{} \\ *}} \mathcal{C}_i = \mathcal{C}_i$*,* $\mathsf{rk}(\mathcal{C}_i = \mathcal{C}_i) < N_0$*,* $\mathsf{rk}(\mathsf{IND}(X)) < N_0$ *and* $\mathsf{rk}(\mathbb{D}) < N_0$*. If* $\delta \in \{0\} \cup \mathsf{Lim}(\Omega)$*, then*

$$\overset{*}{\mathsf{T}}_{q_0}|\mathsf{S} \mathrel{\substack{\delta+n \\ \mid\!\overline{} \\ \mathbb{D}}} \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{\substack{\mathsf{it}(g,\delta+N_0\cdot n) \\ \mid\!\overline{} \\ *}} \Gamma.$$

Proof By induction on the depth α of the derivation. If $\alpha = 0$, then the only interesting case is if Γ contains an $\mathcal{L}^*$-instance of $\forall X \mathsf{IND}(X)$, in which case the claim is by Lemma V.3.18. If α is a limit, then by Lemma V.3.6 we may assume that the last inference was an ω-rule, and the claim follows immediately by the I.H. and the continuity of $\mathsf{it}(g)$. And if $\alpha = \delta{+}n{+}1$ for some $\delta \in \{0\} \cup \mathsf{Lim}(\Omega)$, then the only cases where the I.H. does not apply directly are when $\overset{*}{\mathsf{T}}_{q_0}|\mathsf{S} \mathrel{\substack{\delta+n+1 \\ \mid\!\overline{} \\ \mathbb{D}}} \Gamma$ is obtained from $\overset{*}{\mathsf{T}}_{q_0}|\mathsf{S} \mathrel{\substack{\delta+n \\ \mid\!\overline{} \\ \mathbb{D}}} \Gamma, \neg A$ by a cut. If $A \notin \mathsf{inst}^*(\breve{\mathsf{T}}_{q_0})$, then by I.H., $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{\substack{\mathsf{it}(g,\delta+N_0 n) \\ \mid\!\overline{} \\ *}} \Gamma, \neg A$ and $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \mathrel{\substack{\mathsf{it}(g,\delta+N_0 n) \\ \mid\!\overline{} \\ *}} \Gamma, A$. Since $\mathsf{rk}(A) < N_0$ and $\mathsf{it}(g, \beta{+}1) \geq \omega^{\mathsf{it}(g,\beta)}$ by definition of $\mathsf{it}(g)$, Lemma V.3.15 yields $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n+N_0)}_{*} \Gamma$. If $A \in \mathsf{inst}^*(\forall X \mathsf{IND}(X))$, then $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{<\omega}_{*} A$ by Lemma V.3.18, and by I.H., $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n)}_{*} \Gamma, \neg A$. Since $\mathsf{rk}(A) < N_0$, $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n+N_0)}_{*} \Gamma$ follows as above. Further, if A is $\exists X[X = \mathcal{C}_i]$ for some $i \in I$, then inversion and the I.H. yield $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n)}_{*} \Gamma, U \neq \mathcal{C}_i$, where $U \notin \mathsf{FV}_1(\Gamma, \mathcal{C})$. By Lemma VI.4.8, $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n)}_{*} \Gamma, \mathcal{C}_i \neq \mathcal{C}_i$ follows. Since $\overset{*}{\mathsf{T}}^\epsilon \vdash^{<\omega}_{*} \mathcal{C} = \mathcal{C}$ and $\mathsf{rk}(\mathcal{C}_i = \mathcal{C}_i) < N_0$, $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \vdash^{\mathsf{it}(g,\delta+N_0n+N_0)}_{*} \Gamma$ again follows as above. □

The following corollary is a direct consequence.

Corollary VI.4.10. $\mathsf{it}(g)$ *is a bound of* T_{q_0}.

Eventually, we can prove that g_{x^h} is a sharp bound of T_x. We start with an auxiliary result. Recall that $x^h = x^H$ unless $x = (n{+}1, q_0)$, in which case $x^h = (n{+}2, q_0)$.

Lemma VI.4.11. *Let* $E := \{\alpha : \alpha = \omega^\alpha\}$.

(i) If $x \in Q^* \setminus \{q_0, q_1\}$, *then* $g_{x^h} \subseteq E$.

(ii) If $\mathsf{rk}(\mathbb{C}) < \omega$, *then* $\overset{*}{\mathsf{S}} \vdash^{<g_{x^h}(\alpha)}_{\mathbb{C}} \Gamma \Rightarrow \overset{*}{\mathsf{S}} \vdash^{<g_{x^h}(\alpha)}_{*} \Gamma$.

(iii) For each $x \in Q$ *with* $\deg(x) > 1$, $g_{x^h(\alpha+2)}(0) > g_{x^h(\alpha)}(E(\alpha^+))$.

Proof (i) First, let $y \in Q^H \setminus \{q_0, q_1\}$. We show by a case distinction on y that $H_y \subseteq \mathsf{fix}$. Since $x \in Q^*$ implies $x^h \in Q^H \setminus \{q_0, q_1\}$, we obtain $H_{x^h}(g) \subseteq g' = E$.

If $y = z{+}1$ for some $q_0 \neq z \in Q^H$, then by Lemma III.4.19 (x), $q_1 \rightsquigarrow_r^* y$, and so by Lemma III.5.1, $H_z \subseteq \mathsf{it}$. Further, by Lemma I.3.17, $\mathsf{sh} \circ H_z \subseteq \mathsf{sh} \circ \mathsf{it} = \mathsf{fix}$. As $\mathsf{it} \subseteq \mathsf{sh}$, $H_y = \mathsf{it} \circ H_z \subseteq \mathsf{fix}$ follows. If $o(y) = \gamma$, then by Lemma III.5.3, $f_y = \bigcap_{\alpha<\gamma} f'_{y[\alpha]}$, and as $f_{y[\alpha]} \subseteq f$, also $f'_{y[\alpha]} \subseteq f'$. This shows that $H_y \subseteq \mathsf{fix}$. And if $\deg(y) > 1$ and $o(y) = 1$, then first note that $H_{y[1]} \subseteq \mathsf{fix}$: as $H_{y[0]} \subseteq \mathsf{it}$, we have $\mathsf{sh} \circ H_{y[0]} \subseteq \mathsf{fix}$; further, $H_{y[1]} \subseteq \mathsf{it} \circ H_{y[0]}$ by Lemma III.5.3, and $\mathsf{it} \subseteq \mathsf{sh}$ implies $H_{y[1]} \subseteq \mathsf{sh} \circ H_{y[0]}$. Next, observe that by Lemma III.4.19, $y[1] \rightsquigarrow_r^* y[1{+}\alpha]$, so by Lemma III.5.1, $H_{y[1+\alpha]} \subseteq H_{y[1]}$. Therefore, using Lemma III.5.2, $H_y(f, \alpha) = H_{y[1+\alpha]}(f, 0) \in H_{y[1]}(f) \subseteq \mathsf{fix}(f)$. Thus also in this case, $H_y \subseteq \mathsf{fix}$.

(ii) Immediate by (i) and Lemma V.3.15.

(iii) We show the claim for $y \in Q^H$ with $\deg(y) > 1$ and $o(y) = 1$. Then we observe that if $x \in Q$ with $\deg(x) > 1$, we have $x^H = x^h \in Q^H$, and by Lemma III.4.23, $\deg(x^H) > 1$ and $o(x^h) = 1$.

Let $y \in Q^H$ with $\deg(y) > 1$ and $o(y) = 1$. We have $\mathsf{it}(f, 0) = f(f(0))$, and for $\alpha < \gamma$, $y[\alpha] = y[\gamma][\alpha]$ by Lemma III.4.13. So by Lemma III.5.3, $g_{y[\alpha+1]} \subseteq \mathsf{it}(g_{y[\alpha]})$, in particular, $g_{y[\alpha+1]}(0) \geq g_{y[\alpha]}(g_{y[\alpha]}(0))$. Using Lemma III.5.6, $g_{y[\alpha+1]} \geq \alpha^+$. Hence, $g_{y[\alpha+2]}(0) \geq g_{y[\alpha+1]}(\alpha^+) \geq E(\alpha^+)$ by (i), and $g_{y(\alpha+2)}(0) = g_{y[\alpha+2]+1}(0) \geq g_{y[\alpha+2]}(E(\alpha^+)) > g_{y(\alpha)}(E(\alpha^+))$. □

Next, we recall the following results concerning the interplay of x^h, $x(\alpha)$ and $x[\alpha]$.

(p1) if $\deg(x) = 1$ and $o(x) \in \mathsf{Lim}(\Omega)$, then $x^h = x^H$, so $o(x) = o(x^h)$ by Lemma III.4.23. Further, have that $x^h[\alpha] \rightsquigarrow_r^* (x[\alpha])^h \rightsquigarrow_r^* x^h[\alpha{+}1]$ by Lemma III.4.24, so by Lemma III.5.1, $g_{x^h[\alpha+1]} \subseteq g_{(x[\alpha])^h} \subseteq g_{x^h[\alpha]}$, and by Lemma III.5.3, $g_{x^h} = \bigcap_{\alpha<\gamma} g_{x^h[\alpha]} = \bigcap_{\alpha<\gamma} g_{x[\alpha]^h} = \bigcap_{\alpha<\gamma} g'_{(x[\alpha])^h}$. In particular, for each β and each $\alpha < \gamma$, $g_{(x[\alpha])^h}(g_{x^h}(\beta)) = g_{x^h}(\beta)$.

(p2) If $\deg(x) > 1$, then $x^H[\alpha] \rightsquigarrow_r^* (x(\alpha))^H \rightsquigarrow_r^* x^H[\alpha{+}1]$ by Lemma III.4.24 (ii). Hence by Lemma III.5.1, $g_{z^H[\alpha+1]} \subseteq g_{(z(\alpha))^H}$ and $g_{(z(\alpha))^H} \trianglelefteq g_{z^H[\alpha+1]}$.

Next, we introducing some notations. Then we explain the proof-strategy and provide a further auxiliary result which we use in the proof of the theorem below.

If $x = y{+}1$, we let $\breve{\mathsf{T}}'_x := \mathsf{p}_1(\breve{\mathsf{T}}_y)$, and if $\deg(x) = 1$ and $o(x) = \gamma$, then we let $\breve{\mathsf{T}}'_x := (\forall\xi \lhd \gamma)\mathsf{p}_1(\breve{\mathsf{T}}_{x[\xi]})$. In both cases, we read $\overset{*}{\mathsf{T}}_{x'}|\mathsf{S} \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$ as $\overset{*}{\mathsf{T}}_{x'}|\mathsf{S} \mathrel{\big|\!\frac{<\gamma}{\mathbb{C}}} \Gamma$ for $\mathbb{C} := \mathcal{L}^*_{\Pi^1_0}$, and by Lemma VI.1.3, $\overset{*}{\mathsf{T}}_x|\mathsf{S} \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$ iff $\overset{*}{\mathsf{T}}'_x|\mathsf{S} \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$. In all other cases, we let $\breve{\mathsf{T}}'_x := \breve{\mathsf{T}}_x$.

The proof-strategy is as follows. We will prove by induction on

$$|(x,\alpha)| := \begin{cases} g_{x^h}(\alpha){+}\omega & : \alpha \in \mathsf{Lim}(\Omega) \cup \{0\}, \\ g_{x^h}(\delta^+){+}n & : \text{if } \alpha = \delta{+}n{+}1 \text{ for some } \delta \in \mathsf{Lim}(\Omega) \cup \{0\}, \end{cases}$$

that for each S and each $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}_x|\mathsf{S})$,

(i) $\overset{*}{\mathsf{T}}'_x|\mathsf{S} \mathrel{\big|\!\frac{\alpha}{+}} \Gamma \Rightarrow \overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \mathrel{\big|\!\frac{g_{x^h}(\alpha)}{*}} \Gamma$, if α is a limit,

(ii) $\overset{*}{\mathsf{T}}'_x|\mathsf{S} \mathrel{\big|\!\frac{\alpha}{+}} \Gamma \Rightarrow \overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \mathrel{\big|\!\frac{<g_{x^h}(\alpha^+)}{*}} \Gamma$, if α is not a limit.

Concerning $|(x,\alpha)|$, note that by definition, $|(x,\alpha)| < |(x,\alpha{+}1)|$, and if $\alpha < \gamma$, then $|(x,\alpha)| < |(x,\gamma)|$. Moreover, we have the following, which is exactly what we need to apply the I.H. in the proof of the next theorem.

Lemma VI.4.12. *Assume that $x \in Q$, $\delta \in \mathsf{Lim}(\Omega) \cup \{0\}$ and $n < \omega$.*

(a) If $x = y{+}1$, then $|(y, g_{x^h}(\delta{+}n))| < |(x,\delta{+}1)|$.

(b) If $\deg(x) = 1$ and $\xi < o(x) = \gamma$, then $|(x[\xi]{+}1, g_{x^h}(\delta{+}n))| < |(x,\delta{+}1)|$.

(c) If $\deg(x) > 1$, then $|(x(\delta{+}n), E(\delta^+))| < |(x,\delta{+}1)|$.

Proof (a) Since $\mathsf{fix} = \mathsf{sh} \circ \mathsf{it}$ by Lemma I.3.15, $|(y, g_{x^h}(\delta{+}n))| = g_{y^h}(g_{x^h}(\delta{+}n)){+}\omega < g_{y^h}(g_{x^h}(\delta^+)) = g_{x^h}(\delta^+) = |(x,\delta{+}1)|$. (b) If $\xi < \gamma$, then $|(x[\xi]{+}1, g_{x^h}(\delta{+}n))| < g_{x[\xi]^h+1}(g_{x^h}(\delta{+}n{+}1)) = g_{x^h}(\delta{+}n{+}1) < g_{x^h}(\delta^+) = |(x,\delta{+}1)|$. (c) By Lemma VI.4.11 (iii) and (p2) from page 188, we conclude $|(x(\delta{+}n), E(\delta^+))| < g_{x(\delta+n+1)^h}(E(\delta^+)) < g_{x(\delta+n+3)^h}(0) \leq g_{x^h[\delta+n+4]}(0) < g_{x^h[\delta^+]}(0) = g_{x^h}(\delta^+) = |(x,\delta{+}1)|$. $\square$

Theorem VI.4.13. *For each $x \in Q$, g_{x^h} is a bound of T_x.*

Proof We proof (i) and (ii) on page 188 by a case distinction on x. If $x = q_0$, the claim is by Corollary VI.4.10.

Next, let $x = y{+}1$ for some $y \in Q$. If $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{0}_{+} \Gamma$, then as Γ is arithmetical, already $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{0}_{*} \Gamma$. If $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\alpha}_{+} \Gamma$ is not obtained by a cut, or a cut with an instance of $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S}$, then the claim follows directly by the I.H. Hence, assume that $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\delta+n+1}_{+} \Gamma$ was obtained by a cut from $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\delta+n}_{+} \Gamma, A$ with $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\delta+n}_{+} \Gamma, \neg A$, where A is not an $\mathcal{L}^*$-instance of $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S}$. If A is arithmetical, then by I.H., $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \Gamma, [\neg]A$, and the claim is by Lemma V.3.15 and Lemma VI.4.11. And if A is a relevant $\mathcal{L}^*$-instance of $\mathsf{T}'_x|\mathsf{S}$, say $A_{\mathsf{p}_1(\check{\mathsf{T}}_y)}(Z)$, then $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\delta+n}_{+} \Gamma, \neg A_{\mathsf{p}_1(\check{\mathsf{T}}_y)}(Z)$. Using inversion so that we can apply the I.H., and then undo the inversion step, we obtain $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \Gamma, \neg A_{\mathsf{p}_1(\check{\mathsf{T}}_y)}(Z)$, hence also $\overset{*}{\mathsf{T}}_y|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \Gamma, \neg A_{\mathsf{p}_1(\check{\mathsf{T}}_y)}(Z)$. Now Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_y|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \mathsf{M}_1 \models \Gamma, \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\check{\mathsf{T}}_y)}(Z)$, so Lemma VI.3.16 yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}^2_y|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \mathsf{M}_1 \models \Gamma$. As for $\Delta := \mathsf{M}_1 \models \Gamma$, $\mathsf{M}_1 \models \Gamma$ is $\mathsf{M}_2 \models \Delta$, Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}_y|\mathsf{T}_y|\mathsf{S} \vdash^{g_{x^h}(\delta+m)}_{*} \mathsf{M}_1 \models \Gamma$ for some m. By Lemma VI.4.12 (a), $|(y, g_{x^h}(\delta+m))| < |(x, \delta{+}n{+}1)|$, so the I.H. applies, and we obtain $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_y|\mathsf{S} \vdash^{g_{x^h}(\delta+m+1)}_{*} \mathsf{M}_1 \models \Gamma$, since $g_{y^h}(g_{x^h}(\delta{+}m)) \leq g_{x^h}(\delta{+}m{+}1)$. Again, Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}_y|\mathsf{S} \vdash^{g_{x^h}(\delta+m+1)}_{*} \Gamma$. Another application of the I.H. yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{g_{x^h}(\delta+m+2)}_{*} \Gamma$, that is, $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x^h}(\delta^+)}_{*} \Gamma$.

Next, we consider the case where $\deg(x) = 1$ and $o(x) = \gamma \in \mathsf{Lim}(\Omega)$. We just consider the case where $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\alpha+1}_{+} \Gamma$ is obtained from $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \vdash^{\alpha}_{+} \Gamma, \neg A$ by a cut with some non-arithmetical $A \in \mathsf{inst}^*(\check{\mathsf{T}}'_x|\mathsf{S})$ of the form $\exists X[Z \dot{\in} X \wedge \check{\mathsf{T}}_{x[\xi]}{\restriction}X]$, for some $\xi < \gamma$. Applying $\forall X$-inversion, using the I.H., and undoing the inversion, we obtain $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x^h}(\alpha^+)}_{*} \Gamma, \neg A$. So also $\overset{*}{\mathsf{T}}{}'_{x[\xi]+1}|\mathsf{S} \vdash^{<g_{x^h}(\alpha^+)}_{*} \Gamma$. By Lemma VI.4.12 (b), for each n, $|(x[\xi]{+}1, g_{x^h}(\alpha{+}n))| < |(x, \alpha{+}1)|$, thus the I.H. applies and yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{(x[\xi]+1)^h}(g_{x^h}(\alpha^+))}_{*} \Gamma$, that is $\overset{*}{\mathsf{T}}{}^{\epsilon} \vdash^{<g_{x^h}(\alpha^+)}_{*} \Gamma$.

Finally, we look at the case $\deg(x) = m{+}2$. If $\overset{*}{\mathsf{T}}_x|\mathsf{S} \vdash^{\delta}_{+} \Gamma$, then the claim holds trivially if $\delta = 0$, and if $\delta \in \mathsf{Lim}(\Omega)$, then we may assume that Γ is obtained by an ω-rule and the claim is immediate by I.H. If $\overset{*}{\mathsf{T}}_x|\mathsf{S} \vdash^{\alpha+1}_{+} \Gamma$, then Corollary VI.3.19 yields that $\overset{*}{\mathsf{T}}_{x(\alpha+1)}|\mathsf{S} \vdash^{<\alpha^+} \Gamma$, so $\overset{*}{\mathsf{T}}_{x(\alpha+1)}|\mathsf{S} \vdash^{<E(\alpha^+)}_{*} \Gamma$. By Lemma VI.4.12 (c), $|(x(\alpha{+}1), E(\alpha^+))| < |(x, \alpha{+}1)|$. The I.H. applies and yields $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x(\alpha+1)^h}(E(\alpha^+))}_{*} \Gamma$, hence $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \vdash^{<g_{x^h}(\alpha^+)}_{*} \Gamma$ by Lemma VI.4.11. $\square$

As each bound is also a weak bound, the following is a direct consequence of Theorem VI.4.6.

Corollary VI.4.14. *For each $x \in Q$, g_{x^h} is a sharp bound of T_x.*

We continue by showing some additional, slightly more refined properties of bounds. Most of these properties are already implicit in the proof of Theorem VI.4.13.

Lemma VI.4.15. *Let $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$, $\mathbb{C} = \mathcal{L}^*_{\Pi^1_0}$, $\delta \in \{0\} \cup \mathsf{Lim}(\Omega)$ and f a bound of T_x. Then,*

$$\overset{*}{\mathsf{T}}_{x+1}|\mathsf{S} \overset{\delta+n}{\underset{\mathbb{C}}{\vdash}} \Gamma \implies \overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n)}{\underset{*}{\vdash}} \Gamma.$$

Proof By Lemma VI.1.1, it suffices to show the claim for $\mathsf{p}_1(\overset{*}{\mathsf{T}}_x)|\mathsf{S}$ instead of $\overset{*}{\mathsf{T}}_{x+1}|\mathsf{S}$, which is done by induction on $\alpha = \delta+n$. If $\alpha = 0$, then as Γ is arithmetical, it is already an axiom of $\overset{*}{\mathsf{T}}_x|\mathsf{S}$, hence $\overset{*}{\mathsf{T}}|\mathsf{S} \overset{0}{\underset{\mathbb{C}}{\vdash}} \Gamma$. If α is a limit, then by Lemma V.3.6, we may assume that $\mathsf{p}_1(\overset{*}{\mathsf{T}}_x)|\mathsf{S} \overset{\alpha}{\underset{\mathbb{C}}{\vdash}} \Gamma$ is obtained by an ω-rule, and the claim follows directly from the I.H. and the continuity of $\mathsf{it}(f)$. And if $\alpha = \delta+n+1$, the only interesting case is if Γ was obtained by a cut from $\mathsf{p}_1(\overset{*}{\mathsf{T}}_x)|\mathsf{S} \overset{\delta+n}{\underset{\mathbb{C}}{\vdash}} \Gamma, A$. If A is arithmetical, then by I.H., $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n)}{\underset{*}{\vdash}} \Gamma, \neg A$ and $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n)}{\underset{*}{\vdash}} \Gamma, A$. Using LemmaV.3.15 (ii), and since by Lemma VI.4.11, $\mathsf{it}(f, \delta + 2n{+}2)$ is bigger than the next ε-number above $\mathsf{it}(f, \delta{+}2n)$, we have in particular $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n+2)}{\underset{*}{\vdash}} \Gamma$. And if A is a relevant $\mathcal{L}^*$-instance of $\mathsf{T}|\mathsf{S}$, say $A_{\mathsf{p}_1(\check{\mathsf{T}})}(Z)$, then by the I.H. and Lemma VI.3.14, $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n)}{\underset{*}{\vdash}} \Gamma, \neg A_{\mathsf{p}_1(\check{\mathsf{T}})}(Z)$. Now Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{T}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n)}{\underset{*}{\vdash}} \mathsf{M}_1 \models \Gamma, \neg \mathsf{M}_1 \models A_{\mathsf{p}_1(\check{\mathsf{T}}_x)}(Z)$, so Lemma VI.3.16 yields $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{T}^2_x|\mathsf{S} \overset{<\mathsf{it}(f,\delta+2n)^+}{\underset{\mathbb{C}}{\vdash}} \mathsf{M}_1 \models \Gamma$ for some $\mathbb{C} \subseteq \mathcal{L}^*_{\Pi^1_0}$ with $\mathsf{rk}(\mathbb{C}) < m$ for some m, and $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{T}^2_x|\mathsf{S} \overset{<\mathsf{it}(f,\delta+2n+1)}{\underset{*}{\vdash}} \mathsf{M}_1 \models \Gamma$ follows. As for $\Delta := \mathsf{M}_1 \models \Gamma$, $\mathsf{M}_1 \models \Gamma$ is $\mathsf{M}_2 \models \Delta$, Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}^2_x|\mathsf{S} \overset{<\mathsf{it}(f,\delta+2n+1)}{\underset{*}{\vdash}} \mathsf{M}_1 \models \Gamma$. Since f is a bound of T_x, we obtain $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{T}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n+2)}{\underset{*}{\vdash}} \mathsf{M}_1 \models \Gamma$, and one more application of Lemma VI.3.13 yields $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(f,\delta+2n+2)}{\underset{*}{\vdash}} \Gamma$. □

That $\mathsf{it}(f)$ is a bound of T_{x+1} given that f is a bound of T_x now readily follows.

Lemma VI.4.16. *If f is a bound of T_x, then $\mathsf{it}(f)$ is a bound of T_{x+1}.*

Proof Assume that g_{x^h} is a bound of T_x, and that $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$. By Lemma VI.4.15, we have that $\overset{*}{\mathsf{T}}_{x+1}|\mathsf{S} \overset{<\gamma}{\underset{+}{\vdash}} \Gamma$ entails $\overset{*}{\mathsf{T}}_x|\mathsf{S} \overset{\mathsf{it}(g_{x^h},\beta)}{\underset{*}{\vdash}} \Gamma$ for some $\beta < \gamma$. Since g_{x^h} is a bound of $\overset{*}{\mathsf{T}}_x$, $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \overset{<\mathsf{it}(g_{x^h},\beta+1)}{\underset{*}{\vdash}} \Gamma$ follows, so $\overset{*}{\mathsf{T}}^\epsilon|\mathsf{S} \overset{<\mathsf{it}(g_{x^h},\gamma)}{\underset{*}{\vdash}} \Gamma$. □

The next results tell us how to obtain a bound of T_x if $\deg(x) = 1$ and $o(x) \in \mathsf{Lim}(\Omega)$, given that we have already bounds of the theories $(\mathsf{T}_{x[\alpha]} : \delta \leq \alpha < \gamma)$ for some $\delta < \gamma$. By the above lemma, we then also have bounds of the theories $(\mathsf{T}_{x[\alpha]+1} : \delta \leq \alpha < \gamma)$. The reason for working with the theories $\mathsf{T}_{x[\alpha]+1}$ is that $\deg(x[\alpha]{+}1) = 1$, and thus $\mathsf{inst}^*(\check{\mathsf{T}}_{x[\alpha]+1}) \subseteq \mathcal{L}^*_{\mathsf{e}\Sigma^1_1}$, in fact, each such instance is Σ^1_1.

Lemma VI.4.17. *Assume that* $\deg(x) = 1$ *with* $o(x) \in \mathsf{Lim}(\Omega)$*. Further, assume that for each* $\delta < \xi < o(x)$*,* $g_{x[\xi]^h}$ *is a bound of* $\mathsf{T}_{x[\xi]}$*. Then,* g_{x^h} *is a bound of* T_x*.*

Proof Assume that $\deg(x) = 1$ with $o(x) = \gamma$. By Lemma VI.1.1 (or Lemma VI.1.3), it suffices to show that $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mid\!\frac{\alpha}{\mathbb{C}}\ \Gamma \Rightarrow \overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{g_{x^h}(\alpha)}{*}\ \Gamma$, for each $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$, where $\breve{\mathsf{T}}'_x := (\forall \xi \lhd \gamma)\mathsf{p}_1(\breve{\mathsf{T}}_{x[\xi]})$ and $\mathbb{C} := \mathcal{L}^*_{\Pi^1_0}$. We show this claim by induction on α. The interesting case is if $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mid\!\frac{\alpha+1}{\mathbb{C}}\ \Gamma$ is obtained from $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mid\!\frac{\alpha}{\mathbb{C}}\ \Gamma, \neg A$ by a cut with some non-arithmetical $A \in \mathsf{inst}^*(\breve{\mathsf{T}}'_x|\mathsf{S})$ of the form $\exists X[Z \dot{\in} X \wedge \breve{\mathsf{T}}_{x[\eta]}{\restriction}X]$, for some $\eta < \gamma$. Applying $\forall X$-inversion, using the I.H., and undoing the inversion, we obtain $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{<g_{x^h}(\alpha+1)}{*}\ \Gamma, \neg A$. Now for some $\xi \geq \mathsf{max}(\delta, \eta)$, $\mathsf{p}_1(\overset{*}{\mathsf{T}}_{x[\xi]})|\mathsf{S} \mid\!\frac{<g_{x^h}(\alpha+1)}{\mathbb{C}}\ \Gamma$: as $\eta \leq \xi$, $x[\eta] \rightsquigarrow^* x[\xi]$, thus $\mathsf{T}_{x[\eta]} \vdash \breve{\mathsf{T}}_{x[\xi]}$, and since p_1 is an operation, also $\mathsf{p}_1(\mathsf{T}_{x[\xi]}) \vdash \mathsf{p}_1(\breve{\mathsf{T}}_{x[\eta]})$; further, by Lemma V.4.6, $\mathsf{p}_1(\overset{*}{\mathsf{T}}_{x[\xi]})|\mathsf{S} \mid\!\frac{<\gamma}{\mathbb{C}}\ \Gamma$ implies $\mathsf{p}_1(\overset{*}{\mathsf{T}}_{x[\eta]})|\mathsf{S} \mid\!\frac{<\gamma}{\mathbb{C}}\ \Gamma$. As $g_{(x[\xi]+1)^h}$ is a bound of $\mathsf{T}_{x[\xi]+1}$ (and Lemma VI.1.3), we obtain that $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{<g_{(x[\xi]+1)^h}(g_{x^h}(\alpha+1))}{*}\ \Gamma$, that is $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{<g_{x^h}(\alpha+1)}{*}\ \Gamma$. And if $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mid\!\frac{\alpha+1}{\mathbb{C}}\ \Gamma$ is obtained from $\overset{*}{\mathsf{T}}{}'_x|\mathsf{S} \mid\!\frac{\alpha}{\mathbb{C}}\ \Gamma, [\neg]A$ for some arithmetical A with $\mathsf{rk}(A) = n$, then by I.H., $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{<g_{x^h}(\alpha+1)}{*}\ \Gamma, [\neg]A$, and $\overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \mid\!\frac{<g_{x^h}(\alpha+1)}{*}\ \Gamma$ follows by Lemma VI.4.11 (ii). $\square$

VI.5 Modular ordinal analysis at work yet again

In this final section, we show, dually to what we did in Section III.7, that in some higher type sense, H_{x^H} is a bound of Op_x, and that $H^{+(n+1)}_{x^*}$ is a bound of $\mathsf{Op}^{+(n+1)}_x$ (recall that $x^* := x^H + \mathsf{corr}(x)$; see Definitions III.4.20). We conclude by discussing what meta-theory we implicitly used to proof these results.

$\mathsf{Bd}_n(x)$ is the dual notion of $\mathsf{Prv}_n(x)$, where $\mathsf{Bd}_0(x)$ states that g_{x^h} is a bound of T_x, and $\mathsf{Bd}_{n+1}(x)$ is the corresponding assertion for (higher type) functionals and operations.

Definition VI.5.1. *By* $\mathsf{Bd}_i(x)$ *(*$i \in \mathbb{N}$*) we denote the following statements.*

(i) $\mathsf{Bd}_0(x)$ *says "if* $x \in Q$*, then* g_{x^h} *is a bound of* $\overset{*}{\mathsf{T}}_x$*",*

(ii) $\mathsf{Bd}_1(x)$ *says "for each* y*,* $\mathsf{Bd}_0(y)$ *implies* $\mathsf{Bd}_0(x \circ y)$*",*

(iii) $\mathsf{Bd}_{n+2}(x)$ *says "for each* y*,* $\mathsf{Bd}_{n+1}(1, y)$ *implies* $\mathsf{Bd}_{n+1}(1, x \circ y)$*".*

If $\mathsf{Bd}_0(x)$, we call g_{x^h} a bound of T_x, if $\mathsf{Bd}_1(x)$, we call H_{x^H} a bound of Op_x, and if $\mathsf{Bd}_{n+2}(x)$, then we call $H^{+(n+1)}_{x^*}$ a bound of $\mathsf{Op}^{+(n+1)}_x$. Recall that $x^* := x^H + \mathsf{corr}(x)$ (cf. Definitions III.4.20). The next result corresponds to Lemma III.7.7.

Lemma VI.5.2.

(i) If $x \circ y \in Q$ *and* $\mathsf{Bd}_1(x)$ *and* $\mathsf{Bd}_0(y)$*, then* $\mathsf{Bd}_0(x \circ y)$.

(ii) If $x \circ y \in Q$ *and* $\mathsf{Bd}_{n+2}(x)$ *and* $\mathsf{Bd}_{n+1}(1, y)$*, then* $\mathsf{Bd}_{n+1}(1, x \circ y)$.

(iii) If $x \circ y \in Q$ *and* $\mathsf{Bd}_{n+1}(x)$ *and* $\mathsf{Bd}_{n+1}(y)$*, then* $\mathsf{Bd}_{n+1}(x \circ y)$.

Proof (i) and (ii) are immediate by Definition VI.5.1. (iii) Assume that $x \circ y \in Q$. First, we check the case $n = 0$. Assume $\mathsf{Bd}_1(x)$ and $\mathsf{Bd}_1(y)$. To show: $x \circ y \circ z \in Q$ and $\mathsf{Bd}_0(z)$ yield $\mathsf{Bd}_0(x \circ y \circ z)$. Indeed, $\mathsf{Bd}_1(y)$ and $\mathsf{Bd}_0(z)$ imply $\mathsf{Bd}_0(y \circ z)$, which together with $\mathsf{Bd}_1(x)$ further yields $\mathsf{Bd}_0(x \circ y \circ z)$. Next, we check the case $n > 0$. Assume $\mathsf{Bd}_{n+1}(x)$ and $\mathsf{Bd}_{n+1}(y)$. To show: $x \circ y \circ z \in Q$ and $\mathsf{Bd}_n(1, z)$ yield $\mathsf{Bd}_n(1, x \circ y \circ z)$. Indeed, $\mathsf{Bd}_{n+1}(y)$ and $\mathsf{Bd}_n(1, z)$ implies $\mathsf{Bd}_n(1, y \circ z)$, which together with $\mathsf{Bd}_{n+1}(x)$ further yields $\mathsf{Bd}_n(1, x \circ y \circ z)$. $\square$

In our terminology, the claim of Corollary VI.4.10, that $\mathsf{it}(g)$ is a bound of ACA_0, is now expressed by $\mathsf{Bd}_0(q_0)$. Further, the claim of Lemma VI.4.16, that it is a bound of p_1, reads now $\mathsf{Bd}_1(q_1)$. Observe that these statements correspond to the Lemmas III.7.9 and III.7.11. Moreover, we rephrase (and slightly weaken) Lemma VI.4.17, so that it nicely corresponds to Lemma III.7.10 (ii).

Lemma VI.5.3.

(i) $\mathsf{Bd}_0(q_0)$ *and* $\mathsf{Bd}_1(q_1)$.

(ii) If $\deg(x) = 1$*,* $o(x) = \gamma$ *and* $(\forall \alpha < \gamma)\mathsf{Bd}_0(x[\alpha])$*, then* $\mathsf{Bd}_0(x)$.

Next, we present the Lemma corresponding to Lemma III.7.12.

Lemma VI.5.4. $(\forall x \in Q)[\deg(x) > 1 \land \forall \alpha \mathsf{Bd}_{n+1}(x(\alpha)) \to \mathsf{Bd}_{n+1}(x)]$.

Proof By meta-induction on n. First, we deal with the case $n = 0$. Assume that $x \in Q$ with $\deg(x) > 1$, and $\forall \alpha \mathsf{Bd}_1(x(\alpha))$, and aim for $\mathsf{Bd}_1(x)$. For that, further assume that $\mathsf{Bd}_0(y)$ and $z := x \circ y \in Q$, and aim for $\mathsf{Bd}_0(z)$. To verify $\mathsf{Bd}_0(x \circ y)$, note that $(x \circ y)(\alpha) = x(\alpha) \circ y$ (cf. Lemma III.4.17), and suppose that $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$ and $\overset{*}{\mathsf{T}}_z|\mathsf{S} \,\big|\!\frac{<\gamma}{+}\, \Gamma$. We have to show $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\big|\!\frac{<g_{z^h}(\gamma)}{*}\, \Gamma$.

By Corollary VI.3.19, $\overset{*}{\mathsf{T}}_z|\mathsf{S} \,\big|\!\frac{<\gamma}{+}\, \Gamma$ yields $\overset{*}{\mathsf{T}}_{z(\beta)}|\mathsf{S} \,\big|\!\frac{<\beta^+}{}\, \Gamma$ for a $\beta < \gamma$ (we also used that $\beta' < \beta$ and $\overset{*}{\mathsf{T}}_{z(\beta')}|\mathsf{S} \,\big|\!\frac{<\gamma'}{+}\, \Gamma$ yields $\overset{*}{\mathsf{T}}_{z(\beta)}|\mathsf{S} \,\big|\!\frac{<\gamma'}{+}\, \Gamma$; see the argument given in the middle of the proof of Lemma VI.4.17). Hence we have $\overset{*}{\mathsf{T}}_{z(\beta)}|\mathsf{S} \,\big|\!\frac{<E(\beta^+)}{*}\, \Gamma$. As $\forall \alpha \mathsf{Bd}_1(x(\alpha))$ and $\mathsf{Bd}_0(y)$ by assumption, we obtain $\mathsf{Bd}_0(x(\beta) \circ y)$, that is, $\mathsf{Bd}_0(z(\beta))$. Therefore, using Lemma VI.4.11 (iii), $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\big|\!\frac{<g_{z(\beta+2)^h}(0)}{*}\, \Gamma$. Since by (p2) from page 188, $g_{z(\beta+2)^h}(0) \leq g_{z^h[\beta+3]}(0) < g_{z^h}(\gamma)$, the claim follows.

For the induction step, assume that $n > 0$, and that the claim holds for $n{-}1$. Assume that $x \in Q$ with $\deg(x) > 1$ and $\forall\alpha \mathsf{Bd}_{n+1}(x(\alpha))$, and aim for $\mathsf{Bd}_{n+1}(x)$. For that, further assume that $x \circ y \in Q$ and $\mathsf{Bd}_n(1, y)$, and aim for $\mathsf{Bd}_n(1, x \circ y))$. $\mathsf{Bd}_{n+1}(x(\alpha))$ and $\mathsf{Bd}_n(1, y)$ imply $\mathsf{Bd}_n((1, x(\alpha) \circ y)$, that is, $\mathsf{Bd}_n((1, x \circ y)(\alpha))$, as $\deg(x) \geq 2$ (cf. Lemma III.4.16). Hence we have $\forall\alpha \mathsf{Bd}_n((1, x \circ y)(\alpha))$, and the I.H. yields $\mathsf{Bd}_n(1, x \circ y)$. □

Lemma VI.5.5. *For each n and each $x \in Q$ with* $\deg(x) = 1 \wedge o(x) = \delta_0{+}\gamma$, *and each* $(1, v) \in Q$,

(i) $(\forall\alpha \lhd \gamma)\mathsf{Bd}_{n+1}(x[\delta_0{+}\alpha]) \to \mathsf{Bd}_{n+1}(x) =: C_1(n)$,

(ii) $\mathsf{Bd}_{n+1}(1, v) \to \forall\alpha \mathsf{Bd}_{n+1}(1{+}\alpha, v) =: C_2(n)$.

Proof First note that (ii) follows using (i) by induction on α, as (i) settles the limit case. Hence, it suffices to show (i), which is done by meta-induction on n. First, we look at the case $n = 0$.

Assume $x \in Q$ with $\deg(x) = 1$, $o(x) = \delta_0{+}\gamma$, and $(\forall\alpha{\lhd}\gamma)\mathsf{Bd}_1(x[\delta_0{+}\alpha])$, and aim for $\mathsf{Bd}_1(x)$. For that, further assume that $\mathsf{Bd}_0(y)$ and $z := x{\circ}y \in Q$, and aim for $\mathsf{Bd}_0(z)$. Thereto, let δ_1 so that for each β, $x[\beta] \circ y = z[\delta_1{+}\beta]$, and so $o(z) = \delta_1{+}\delta_0{+}\gamma$ (cf. Lemma III.4.16). As for each $\alpha < \gamma$, $\mathsf{Bd}_1(x[\delta_0{+}\alpha])$, we also have $\mathsf{Bd}_0(x[\delta_0{+}\alpha] \circ y)$, that is, $\mathsf{Bd}_0(z[\delta_1{+}\delta_0{+}\alpha])$. Therefore, $\mathsf{Bd}_0(z)$ follows from Lemma VI.4.17.

Next, we consider the induction step. It is assumed that $n > 0$, and that (i) and (ii) hold for $n{-}1$ ($C_1(n{-}1)$ and $C_2(n{-}1)$). We show that (i) holds for n, i.e. $C_1(n)$.

Assume that $x \in Q$ with $\deg(x) = 1$ and $o(x) = \delta_0{+}\gamma$ and $(\forall\alpha \lhd \gamma)\mathsf{Bd}_{n+1}(x[\delta_0{+}\alpha])$, and aim for $\mathsf{Bd}_{n+1}(x)$. For that, further assume that $z := (1, x{\circ}y) \in Q$ and $\mathsf{Bd}_n(1, y)$, and aim for $\mathsf{Bd}_n(z)$. Thereto, let δ_1 so that for each β, $x[\beta] \circ y = (x \circ y)[\delta_1{+}\beta]$ and $o(z) = o(x \circ y) = \delta_1{+}\gamma$ (cf. Lemma III.4.16). Recall that by Lemma III.4.12 (iv), $(1, (x{\circ}y)^-)[\alpha] = (1, (x{\circ}y)[\alpha])$. The assumptions $(\forall\alpha{\lhd}\gamma)\mathsf{Bd}_{n+1}(x[\alpha])$ and $\mathsf{Bd}_n(1, y)$ yield for each $\alpha < \gamma$, $\mathsf{Bd}_n(1, x[\alpha] \circ y)$, that is $\mathsf{Bd}_n((1, (x \circ y)^-)[\delta_1{+}\alpha])$. Hence the I.H. yields $\mathsf{Bd}_n(1, (x \circ y)^-)$. So by (ii), $\forall\beta \mathsf{Bd}_n(1{+}\beta, (x \circ y)^-)$, that is, $\forall\beta \mathsf{Bd}_n(z(\beta))$. Finally, $\mathsf{Bd}_n(z)$ is by Lemma VI.5.4. □

Lemma VI.5.6. *For all n,* $\mathsf{Bd}_{n+1}(q_1)$.

Proof By meta-induction on n. The case $n = 0$ is by Lemma VI.5.3 (i). For the induction step, assume that the claim holds for n. To show $\mathsf{Bd}_{n+2}(q_1)$, assume that $y{+}1 \in Q$ and $\mathsf{Bd}_{n+1}(1, y)$, and aim for $\mathsf{Bd}_{n+1}(x)$ for $x := (1, y{+}1)$. Note that $\deg(x) > 1$. Once we know that $\forall\alpha \mathsf{Bd}_{n+1}(x(\alpha))$, the claim is by Lemma VI.5.4. By Lemma VI.5.5 (ii), $\mathsf{Bd}_{n+1}(1, y)$ yields $\forall\alpha \mathsf{Bd}_{n+1}(1{+}\alpha, y)$. Since $x(\alpha) = x[\alpha]{+}1 =$

$(1{+}\alpha, y){+}1$, and $\mathsf{Bd}_{n+1}(1{+}\alpha, y)$ and $\mathsf{Bd}_{n+1}(q_1)$ yield $\mathsf{Bd}_{n+1}((1{+}\alpha, y){+}1)$, we also have $\forall\alpha \mathsf{Bd}_{n+1}(x(\alpha))$. □

Putting the pieces together yields a proof of the main result of part II.

Theorem VI.5.7. *For each $x \in Q$ and each $m > 0$, $\mathsf{Bd}_m(x)$.*

Proof Fix N_0 so that $x \in Q_{N_0+1-m}$. We show by meta-induction on $n < N_0$, that $(\forall x \in Q_{n+1})\mathsf{Bd}_{N_0-n}(x)$. For $n := N_0 - m$, $\mathsf{Bd}_m(x)$ then follows.
If $n = 0$, then $\mathsf{Bd}_{N_0}(q_0)$ holds trivially, and as we have $\mathsf{Bd}_{N_0}(q_1)$ by Lemma VI.5.6, $\mathsf{Bd}_{N_0}(1{+}\beta, q_0)$ is by Lemma VI.5.5 (ii). Therefore, $(\forall x \in Q_1)\mathsf{Bd}_{N_0}(x)$.

For the induction step, assume $n{+}1 < N_0$ and $(\forall x \in Q_{n+1})\mathsf{Bd}_{N_0-n}(x)$. We show $(\forall x \in Q_{n+2})\mathsf{Bd}_{N_0-n-1}(x)$ by induction on $\rightsquigarrow^*\restriction Q_{n+2}$. We consider the following possible cases.

(i) $x = y{+}1$. If $x = q_1$, $\mathsf{Bd}_{N_0-n-1}(q_1)$ is by Lemma VI.5.6. Else, we have $\mathsf{Bd}_{N_0-n-1}(y)$ by I.H. Together with $\mathsf{Bd}_{N_0-n-1}(q_1)$, this yields $\mathsf{Bd}_{N_0-n-1}(x)$.

(ii) $\deg(x) = 1 \wedge o(x) = \gamma$. By I.H., $(\forall\alpha \lhd \gamma)\mathsf{Bd}_{N_0-n-1}(x[\alpha])$, and $\mathsf{Bd}_{N_0-n-1}(x)$ is by Lemma VI.5.5 (i).

(iii) $\deg(x) > 1$. Then there are y, z with $\deg(y) > 0$ so that $x =_{NF} (1, y) \circ z$. As $z \rightsquigarrow^* x$ by Lemma III.4.19 (iv), the I.H. yields $\mathsf{Bd}_{N_0-n-1}(z)$. As $(1, y) \in Q_{n+2}$, $y \in Q_{n+1}$, and meta-by I.H., $\mathsf{Bd}_{N_0-n}(y)$. Together with $\mathsf{Bd}_{N_0-n-1}(1, q_0)$ we obtain $\mathsf{Bd}_{N_0-n-1}(1, y)$, and $\mathsf{Bd}_{N_0-n-1}(1, y)$ and $\mathsf{Bd}_{N_0-n-1}(z)$ finally imply $\mathsf{Bd}_{N_0-n-1}(x)$.

□

Since $\mathsf{Bd}_1(x)$ and $\mathsf{Bd}_0(q_0)$ yield $\mathsf{Bd}_0(x)$, we have the next corollary.

Corollary VI.5.8. $(\forall x \in Q)\mathsf{Bd}_0(x)$.

Furthermore, we obtain that for each composite name c, g^{c^h} is a sharp bound of T^c.

Corollary VI.5.9. *For each composite name c, g^{c^h} is a sharp bound of T^c.*

Proof That T^c proves g^{c^h} is by Lemma V.1.9. That g^{c^h} is a bound of T^c is immediate by induction on the length of the composite name c, using that if f_1 is a bound of T^c and f_2 is a bound of T_x, then $f_2 \circ f_1$ is a bound of $\mathsf{T}^{(x,c)}$: if $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}^{(x,c)})$ and $\overset{*}{\mathsf{T}}{}^{(x,c)} \mathrel{\Big|\!\frac{<\gamma}{+}} \Gamma$, then $\overset{*}{\mathsf{T}}{}^{\epsilon}|\overset{*}{\mathsf{T}}{}^{c} \mathrel{\Big|\!\frac{<f_2(\gamma)}{*}} \Gamma$, so by Lemma V.3.10 $\overset{*}{\mathsf{T}}{}^{\epsilon}|\overset{*}{\mathsf{T}}{}^{c} \mathrel{\Big|\!\frac{<f_2(\gamma)}{*}} \mathsf{M}_0 \models \Gamma$, and further, by Lemma VI.3.13, $\overset{*}{\mathsf{T}}{}^{c} \mathrel{\Big|\!\frac{<f_2(\gamma)}{*}} \Gamma$; the claim now follows by the assumption that f_1 is a bound of T^c. □

In particular, we have obtained the results announced in the introduction. For theories of the form $\mathsf{T}_x + (\mathsf{I_N})$, observe that by Lemma V.3.18 (i), $\overset{*}{\mathsf{T}}{}^{\epsilon} \mathrel{\Big|\!\frac{<\omega}{*}} A$ for each instance A of $(\mathsf{I_N})$. Therefore, $\mathsf{T}_x + (\mathsf{I_N}) \vdash \Gamma$ implies $\overset{*}{\mathsf{T}}_x \mathrel{|\!\!\frac{<\omega}{}} \Gamma$, which by Lemma V.3.15 yields $\overset{*}{\mathsf{T}}_x \mathrel{|\!\!\frac{<\varepsilon_0}{*}} \Gamma$. Using that g_{x^h} is a bound of T_x, the Boundedness Lemma implies $|\mathsf{T}_x + (\mathsf{I_N})| \leq g_{x^h}(\varepsilon_0)$.

The presentation of the ordinals in the form $\varphi\vec{\alpha}$ is due to Definition IV.5.14 and Corollary IV.5.16, and the presentation of the ordinals in the form $\vartheta\gamma$ is due to Corollary IV.5.13.

Corollary VI.5.10. *Let* $\Omega_0 := 1$, $\Omega_{n+1} := \Omega^{\Omega_n}$, $\Omega_0(\alpha) := \alpha$, *and* $\Omega_{n+1}(\alpha) := \Omega^{\Omega_n(\alpha)}$.

(i) $|\mathsf{ACA_0}| = \varepsilon_0$ *and* $|\mathsf{ACA}| = \varphi 1 \varepsilon_0$.

(ii) $|\mathsf{p}_1(\mathsf{ACA_0})| = \varphi 20$ *and* $|\mathsf{p}_1(\mathsf{ACA_0}) + (\mathsf{I_N})| = \varphi 2 \varepsilon_0$.

(iii) $|\Sigma^1_1\text{-}\mathsf{DC_0}| = \varphi\omega 0$ *and* $|\Sigma^1_1\text{-}\mathsf{DC}| = \varphi\varepsilon_0 0$.

(iv) $|\mathsf{ATR_0}| = \Gamma_0$ *(Feferman-Schütte ordinal), and* $|\mathsf{ATR}| = \varphi 10 \varepsilon_0$.

(v) $|\mathsf{ATR_0} + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi 1 \omega 0$ *and* $|\mathsf{ATR} + (\Sigma^1_1\text{-}\mathsf{DC})| = \varphi 1 \varepsilon_0 0$.

(vi) $|\Sigma^1_1\text{-}\mathsf{TDC_0}| = \varphi\omega 00$ *and* $|\Sigma^1_1\text{-}\mathsf{TDC}| = \varphi\varepsilon_0 00$.

(vii) $|\mathsf{p}_1(\Sigma^1_1\text{-}\mathsf{TDC_0})| = \varphi 1000$ *(Ackermann ordinal).*

(viii) $|\mathsf{p}_2^{n+2}(\mathsf{ACA_0})| = \varphi\omega \underbrace{0\ldots0}_{n} 0$, $|\mathsf{p}_2^{n+2}(\mathsf{ACA_0}) + (\mathsf{I_N})| = \varphi\varepsilon_0 \underbrace{0\ldots0}_{n} 0$ *and*

$|\mathsf{p}_1\mathsf{p}_2^{n+2}(\mathsf{ACA_0})| = \varphi 1 \underbrace{0\ldots0}_{n+1} 0$.

(ix) $|\mathsf{p}_3(\mathsf{ACA_0})| = \vartheta\Omega^\omega$ *(small Veblen number), and* $|\mathsf{p}_1\mathsf{p}_3(\mathsf{ACA_0})| = \vartheta\Omega^\Omega$ *(big Veblen number).*

(x) $|\mathsf{p}_{n+3}(\mathsf{ACA_0})| = \vartheta\Omega_n(\omega)$, $|\mathsf{p}_{n+3}(\mathsf{ACA})| = \vartheta\Omega_n(\varepsilon_0)$ *and* $\mathsf{p}_1|\mathsf{p}_{n+3}(\mathsf{ACA})| = \vartheta\Omega_{n+1}$.

VI.5.1 In which meta-theory did we prove $(\forall x \in Q)\mathsf{Bd}_0(x)$?

To conclude, we provide a rough sketch of how to prove a formalized and restricted version of the statement "g_{x^h} is a bound of T_x" in $\mathsf{ACA_0}$ plus some amount of transfinite induction. To do so, we first of all need a way to express $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mathrel{|\!\!\frac{\alpha}{+}} \Gamma$, which is achieved by assigning a code $\ulcorner\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mathrel{|\!\!\frac{\alpha}{+}} \Gamma\urcorner$ to such a derivation. $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mathrel{|\!\!\frac{\alpha}{+}} \Gamma$ is then expressed by $\ulcorner\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mathrel{|\!\!\frac{\alpha}{+}} \Gamma\urcorner \in D$, where D is the set of codes of all such derivations.

The set D of codes of such derivations is specified by a positive inductive definition, that is, the least fixed point of $X \mapsto \{x : A(X, x)\}$, where $A(U, u)$ is an arithmetical formula that contains U only positively. Actually, it is not required that D is the least fixed point, we just need that D is a fixed point. In order for D to be a set, provably in ACA_0, we further have to arrange things so that D is a fixed point of a positive Π^0_1 inductive definition $A(U,u)$. In this case, assuming that $\pi^0_1(U,x,y,e)$ is a universal Π^0_1-formula so that for each Π^0_1-formula $B(U,x,y)$, there is an index e_B so that $\mathsf{ACA}_0 \vdash \forall X,x,y[B(X,x,y) \leftrightarrow \pi^0_1(X,x,y,\overline{e}_B)]$, it is readily checked that for an index e so that $A(\{z : \pi^0_1(U,z,y,y)\},x)$ iff $\pi^0_1(U,x,y,\overline{e})$, $D := \{x : \pi^0_1(\emptyset,x,\overline{e},\overline{e})\}$ is the desired Π^0_1-fixed point.

A straightforward coding of derivations leads however to a positive Π^0_1-$\vee$-Σ^0_1 inductive definition, as e.g. in case of an $\exists x$-rule, we need to guess a witness. Therefore, we proceed as detailed in Schwichtenberg [23] section 4.2.2, and use codes of derivation which provide as additional information codes of immediate subderivations, or more to the point, a recursive function, that computes these codes. A code of $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mid\!\frac{\alpha}{+}\, \Gamma$ is then a tuple of the form $\langle e,x,c,r,\alpha,+,\Gamma\rangle$, where $\{e\}(i)$ computes a code of the ith premise of the last inference which is coded by r; further we wrote Γ instead of $\ulcorner\Gamma\urcorner$.

We just hint at how the clauses of such an inductive definition look like for the ω-rule and the $\exists x$-rule. Assume that 0 is a code of the ω-rule, and 1 a code for the $\exists x$-rule. Then, $\langle e,x,c,0,\alpha,+,\ulcorner\Gamma,\forall x A(x)\urcorner\rangle \in D$, if for each n, $\{e\}(n) \in D$ and $\{e\}(n)$ is of the form $\langle e',x,c,r,\beta,+,\ulcorner\Gamma,A(\overline{n})\urcorner\rangle$, where $\beta \lhd \alpha$. And accordingly, $\langle e,x,c,1,\alpha,+,\ulcorner\Gamma,\exists x A(x)\urcorner\rangle \in D$, if $\{e\}(0) \in D$ and $\{e\}(0)$ is of the form $\langle e',x,c,r',\beta,+,\Gamma,A(s)\rangle$, where $\beta \lhd \alpha$.

Of course, we have to check that the previously shown reduction properties also hold for derivations coded by D which store additional information, in particular, we would have to show that we can always provide indices of recursive functions that compute the respective subderivations. More precisely, we would have to provide recursive functions $t_1(d)$, $t_2(d)$ and $t_3(d)$ that compute explicitly the proof-transformations given in the Lemma VI.3.13, Lemma VI.4.15 and Corollary VI.3.19: if d codes the derivation $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_x|\mathsf{T}^c \mid\!\frac{\alpha}{*}\, \mathsf{M}_0 \models \Gamma$, then $t_1(d)$ codes the derivation $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mid\!\frac{\alpha}{*}\, \Gamma$, if d codes the derivation $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_x|\mathsf{T}^c \mid\!\frac{\alpha}{+}\, \Gamma$, then $t_2(d)$ codes the derivation $\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}^2_x|\mathsf{T}^c \mid\!\frac{<\alpha^+}{*}\, \mathsf{M}_1 \models \Gamma$, and if d codes the derivation $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \mid\!\frac{\alpha}{*}\, \mathsf{M}_0 \models \Gamma$, then $t_3(d)$ codes the derivation $\overset{*}{\mathsf{T}}_{x(\alpha)}|\mathsf{T}^c \mid\!\frac{\alpha^+}{+}\, \Gamma$. A recursive definition of these function $t_i(d)$ $(i \in \{1,2,3\})$ can be read off from the proofs of the respective results. The recursion theorem then provides us with an index of t_i; that t_i has the right properties is then proved using transfinite induction on α. As we have to show that for each code of a derivation $d \in D$ with depth below α, there is a computation of $t_i(d)$ that yields a code of a derivation $d' \in D$, transfinite induction for all sets that are Π^0_2 in D is required, that is, $\forall z \mathsf{TI}_{\lhd}((D')_z,\alpha)$, where $D' := \{\langle x,e\rangle : \pi^0_2(D,x,e)\}$ for some universal Π^0_2-formula $\pi^0_2(U,x,e)$.

Furthermore, it turns out that $\forall z \mathsf{TI}_{\lhd}((D')_z, \lambda)$ is what we need besides ACA_0 to prove a formalized version of Theorem VI.4.13, that for each name x, g_{x^h} is a bound of T_x, given that the resulting derivation in $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{T}^c$ has depth at most λ. That is, if we let $\mathsf{Bd}_0^{\lambda}(x) := \forall \gamma[g_{x^h}(\gamma) \trianglelefteq \lambda \to \mathsf{Bd}_0'(D, x, \gamma)]$, where $\mathsf{Bd}_0'(D, x, \gamma)$ expresses that for each $d \in D$ that codes $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \,\big|\!\frac{<\gamma}{+}\, \Gamma$, there exists a $d' \in D$ that codes $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{T}^c \,\big|\!\frac{<g_{x^h}(\gamma)}{*}\, \Gamma$, then

$$\mathsf{ACA}_0 + \forall z \mathsf{TI}_{\lhd}((D')_z, \lambda) \vdash (\forall x \in Q)\mathsf{Bd}_0^{\lambda}(x).$$

Finally, to obtain a formalized version of Theorem VI.5.7, that for each $x \in Q$ and each $m > 0$, $\mathsf{Bd}_m(x)$, we let $\mathsf{Bd}_1^{\lambda}(x) := \forall y[\mathsf{Bd}_0^{\lambda}(y) \to \mathsf{Bd}_0^{\lambda}(x \circ y)]$, and for each $n > 0$, $\mathsf{Bd}_{n+1}^{\lambda}(x) := \forall y[\mathsf{Bd}_n^{\lambda}(1, y) \to \mathsf{Bd}_n^{\lambda}(1, x \circ y)]$. So $\mathsf{Bd}_{n+1}^{\lambda}(x)$ are modified versions of $\mathsf{Bd}_{n+1}(x)$. These modifications are justified by the observation that the successive reduction of a derivation $\overset{*}{\mathsf{T}}_x|\mathsf{T}^c \,\big|\!\frac{<\gamma}{+}\, \Gamma$ to a derivation $\overset{*}{\mathsf{T}}^{\epsilon}|\mathsf{T}^c \,\big|\!\frac{<\gamma}{*}\, \Gamma$ only relies on intermediate derivations whose depths are smaller than $g_{x^h}(\gamma)$. Hence, we have for each n, that

$$\mathsf{ACA}_0 + \mathsf{Wo}_{\lhd}(\lambda) \vdash (\forall x \in Q)\mathsf{Bd}_n^{\lambda}(x).$$

Conclusion

In this thesis, we managed to compute sharp bounds of the theories $(\mathsf{Op}_x(\mathsf{ACA}_0) : x \in Q)$ – in particular their proof-theoretic ordinals – using predicative methods only. Therefore, the theories $(\mathsf{Op}_x(\mathsf{ACA}_0) : x \in Q)$ are meta-predicative. This confirms that ordinal analysis with predicative methods allows to handle theories of ordinal strength up to the Bachmann-Howard ordinal.

The essence of predicative methods is often described as "bottom-up": ordinal notations are constructed by adding new terms to build additional notations whenever closure w.r.t. the previously introduced terms is reached. With regard to ordinal notation systems, the "bottom-up" approach mainly accounts for the interpretation of the notations, the elements of a primitive recursive well-ordering: whether a code (x, α) is interpreted bottom-up as $H_x(g, \alpha)$, or rather top-down as the collapse $\vartheta(\|x\|'+\alpha)$ (cf. Theorem IV.5.12). However, the "bottom-up" approach is visible more clearly when considering the infinitary systems employed to eliminate cuts.

In our view, the characteristic feature of predicative methods is that the soundness of the rules of these infinitary systems is self-evident and depends only on the rules' premises; that is, $\overset{*}{\mathsf{T}} \,\big|\!\frac{\alpha}{}\, \Gamma$, if there is a rule, so that all premises $(\Gamma_i : i \in I)$ of this rule are derived with depth $\alpha_i < \alpha$, where the depth is just the height of the proof-tree. In fact, the infinitary systems $(\overset{*}{\mathsf{T}}_x : x \in Q)$ used in this thesis canonically correspond to the formal theories $(\mathsf{T}_x : x \in Q)$. Therefore, all rules of $\overset{*}{\mathsf{T}}_x$ are trivially sound, and partial cut-elimination for $\overset{*}{\mathsf{T}}_x$ is obtained straightforwardly. This is in clear contrast to the trademarks of impredicative methods: infinitary systems whose soundness also depends on complex proof-transformation properties, as is the case for infinitary systems equipped with the Ω-rule, where moreover, the height of the proof-tree is measured w.r.t. an ordering that involves uncountable ordinals and collapsing functions.

In the following, we reflect on how we obtained our main results and how the proofs that $(\forall x \in Q)\mathsf{Bd}_0(x)$, and that for each n, $(\forall x \in Q)\mathsf{Bd}_n(x)$ given in the second part relate to the proofs of $\mathsf{T}^\epsilon \vdash (\forall x \in Q^*_{N_0})\mathsf{Prv}_0(x)$, and that for all $n < N_0$, $\mathsf{T}^\epsilon \vdash (\forall x \in Q_{n+1})\mathsf{Prv}_{N_0-n}(x)$, given in the first part, where N_0 is some fixed but aritrary large number.

Underlying ideas and concepts

At the base of our modular ordinal analysis lie the sets Q^H and Q of names to address all required functionals and operations, which are constructed by iterated transfinite composition from the basic functionals $(\mathsf{It}_{n+1} : n \in \mathbb{N})$ (where $\mathsf{it} := \mathsf{It}_1$ and $\mathsf{It} := \mathsf{It}_2$) and the basic operations $(\mathsf{p}_{n+1} : n \in \mathbb{N})$ (see Definitions III.2.1 and III.3.1).

With functionals, we assigned to each $x \in Q^H \setminus \{q_0\}$ a functional H^{+n}_x of type $n{+}2$, so that $H^{+n}_{(\alpha,q_0)} = \mathsf{It}^\alpha_{n+1}$, and if $x \neq q_0$, then $H^{+n}_{(\beta,x)} = (H^{+(n+1)}_x(\mathsf{It}_{n+1}))^\beta$, and $H^{+n}_{\langle x_1,\ldots,x_k\rangle} := H^{+n}_{x_1} \circ \ldots \circ H^{+n}_{x_k}$. Recall that if $\langle x_1, \ldots, x_k\rangle \in Q^H$, then $x_i = (\alpha_i, y_i)$ and $y_1 <^H \ldots <^H y_k$. Furthermore, for $x = L(y_0 \circ_1 \ldots \circ_1 y_m) \in Q^H$ with $\mathsf{lh}(y_i) = 1$ $(0 \le i \le m)$,

$$H^{+n}_x = (H^{+(n+m)}_{y_0}, H^{+(n+m-1)}_{y_1}, \ldots, H^{+(n+0)}_{y_m}).$$

With operations, the situation is more delicate: for instance, different names are required for the operations $\forall n \mathsf{p}_1\mathsf{p}^n_2\mathsf{p}_1$ and $\mathsf{p}^\omega_2\mathsf{p}_1$. We used $(1, (\omega, q_0)^-)$ as name for the former and $(1, (\omega, q_0))$ as name for the latter, where $(\omega, q_0)^-$ is a prename. The main difference between Q^H and Q shows in names with $\deg(x) = 1$ and $o(x) \in \mathsf{Lim}(\Omega)$: if $x \in Q^H$, then $(x)_0$ is of the form (γ, z), while if $x \in Q$, then $(x)_0$ may also be of the form $(1{+}\alpha, z^-)$.

The operations $(\mathsf{Op}^{+n}_x : x \in Q^*)$ are defined by recursion on $\rightsquigarrow^*$, the transitive closure of $y \rightsquigarrow x :\Leftrightarrow (\exists \alpha < o(x))(y = x[\alpha])$:

$$\mathsf{Op}^{+n}_{q_1} :\Leftrightarrow \mathsf{p}_{n+1}, \text{ and if } \deg(x) = m{+}1, \text{ then } \mathsf{Op}^{+n}_x :\Leftrightarrow (\forall \alpha \lhd o(x))(\mathsf{p}_{m+n+1} \circ \mathsf{Op}^{+n}_{x[\alpha]}).$$

We point out that the ordertype of $\rightsquigarrow^*\restriction x$ is only about the size of the largest ordinal $|x|$ occurring in x^H (see Definition III.5.5): indeed, if $\mathsf{lv}(x) = n$, then the ordertype of $\rightsquigarrow^*\restriction\{y : y \rightsquigarrow^* x\}$ is less than $(|x^H| \cdot \omega)^n$.

Similarly to functionals, we have that for $x = L(y_0 \circ_1 \ldots \circ_1 y_m) \circ z \in Q^*$,

$$\mathsf{Op}_x^{+n} \Leftrightarrow (\mathsf{Op}_{y_0}^{+(n+m)} \circ \mathsf{Op}_{y_1}^{+(n+m-1)} \circ \ldots \circ \mathsf{Op}_{y_m}^{+n}) \circ \mathsf{Op}_z^{+n}.$$

The main ideas of modular ordinal analysis are that we can adequately describe a theory T by a sharp bound f (T proves f, and conversely, f is a bound of T), and accordingly, that each operation Op_x can be adequately described by a corresponding functional H_{x^H}: if f is a sharp bound of T, then $H_{x^H}(f)$ is a sharp bound of $\mathsf{Op}_x(\mathsf{T})$. We fixed $\mathsf{T} := \mathsf{ACA}_0$ and $g(\alpha) := \omega^{1+\alpha}$, and observed that indeed for each name $x \in Q$, $g_{x^h} := H_{1+x^H}(g)$ is a sharp bound of $\mathsf{T}_x := \mathsf{Op}_x(\mathsf{T})$.

Technically, T_x proves g_{x^h} is defined as

$$\mathsf{Prv}_0(x) := \breve{\mathsf{T}}_x \to \forall\alpha[\mathsf{Wo}_\triangleleft(\alpha) \land \mathsf{TI}_\triangleleft(\mathcal{C}_x, \alpha) \to \mathsf{Wo}_\triangleleft(g_{x^h}(\alpha))],$$

where $\mathcal{C}_x$ is a suitable class term (cf. Definition III.7.2). On the other hand, that g_{x^h} is a bound of T_x, or $\mathsf{Bd}_0(x)$ for short, states that if $\deg(x) = m$, then for each S, each $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m\dot{-}1}}(\mathsf{T}_x|\mathsf{S})$ and each $\gamma \in \mathsf{Lim}(\Omega)$,

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\frac{\;<\gamma\;}{+}\, \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^\epsilon|\mathsf{S} \,\frac{\;<f(\gamma)\;}{*}\, \Gamma.$$

Recall that $\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\frac{<\gamma}{+}\, \Gamma$ indicates that the cut-rule is restricted to formulas in $\mathsf{inst}^*(\breve{\mathsf{T}}_x|\mathsf{S})$ and some additional formulas that do not impede the cut-elimination process; these additional cuts can be eliminated cheaply at a later stage (cf. Definition VI.4.1).

Higher type variants of these notions read as follows: we have that Op_x proves H_{x^H} iff $\mathsf{Prv}_1(x) := \forall y[\mathsf{prv}_0(y) \to \mathsf{Prv}_0(x \circ y)]$, and accordingly, H_{x^H} is a bound of Op_x iff $\mathsf{Bd}_1(x) := \forall y[\mathsf{Bd}_0(y) \to \mathsf{Bd}_0(x \circ y)$. Further, $\mathsf{Op}_x^{+(n+1)}$ proves $H_{x^*}^{+(n+1)}$, if $\mathsf{Prv}_{n+2}(x) := \forall y[\mathsf{prv}_{n+1}(1, y) \to \mathsf{Prv}_{n+1}(1, x \circ y)]$, and $H_{x^*}^{+(n+1)}$ is a bound of $\mathsf{Op}_x^{+(n+1)}$ if $\mathsf{Bd}_{n+2}(x) := \forall y[\mathsf{Bd}_{n+1}(1, y) \to \mathsf{Bd}_{n+1}(1, x \circ y)]$, where $x^* := x^H + \mathsf{corr}(x)$ (cf. Definition III.4.20).

Reviewing the proof strategies of the main results

In the first part, we showed $\overset{*}{\mathsf{T}}{}^\epsilon \vdash (\forall x \in Q_{N_0})\mathsf{Prv}_0(x)$, and $\overset{*}{\mathsf{T}}{}^\epsilon \vdash (\forall x \in Q^*_{N_0})\mathsf{Prv}_{n+1}(x)$ for each $n < N_0$. In the second part, we then computed bounds of the theories $(\mathsf{T}_x : x \in Q)$: we provided two different proof of $(\forall x \in Q)\mathsf{Bd}_0(x)$ that differed mainly w.r.t. the ordering used for the induction which has consequences mainly for names x with $\deg(x) > 1$. A direct proof transformed a derivation $\mathsf{T}_x \,\frac{<\gamma}{+}\, \Gamma$ with $\deg(x) > 1$ into a derivation $\mathsf{T}_{x(\beta)} \,\frac{<E(\gamma)}{*}\, \Gamma$ for some $\beta < \gamma$, which by the I.H.

could further be transform into a cut-free derivation. And the second proof made use of higher type bounds: in order to show $\mathsf{Bd}_1(x)$, we first proved a stronger result, namely that for each n, $\mathsf{Bd}_{n+1}(x)$, using that a name x with $\deg(x) = m{+}2$, is of the form $x =_{NF} (1, y) \circ z$ for some y, z with $\deg(y) = m{+}1$, and then using that by I.H., $\mathsf{Bd}_{n+2}(y)$ and $\mathsf{Bd}_{n+1}(z)$, which then yielded $\mathsf{Bd}_{n+1}(x)$. Since $\mathsf{Bd}_0(q_0)$, and thus $\mathsf{Bd}_1(x)$ yields $\mathsf{Bd}_0(x)$, we also have $(\forall x \in Q)\mathsf{Bd}_0(x)$.

A direct proof of $(\forall x \in Q)\mathsf{Bd}_0(x)$ was possible since we worked in a meta-theory and assumed a reasonable amount of transfinite induction. More precisely, we showed that for each arithmetical sequent Γ,

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\frac{<\gamma}{+}\, \Gamma \Rightarrow \overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{S} \,\frac{<g_{x^h}(\gamma)}{*}\, \Gamma,$$

by induction on $|(x,\gamma)|$ (cf. page 188), using the following reduction properties.

(i) If $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\Pi^1_0}(\mathsf{T}|\mathsf{S})$, then

$$\overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}_y|\mathsf{S} \,\frac{\alpha}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_1}}\, \mathsf{M}_1 \models \neg A_{\mathsf{p}_1(\breve{\mathsf{T}}_y)}(Z), \mathsf{M}_1 \models \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}{}^{\epsilon}|\mathsf{T}^2_y|\mathsf{S} \,\frac{<\alpha^+}{\mathcal{L}^*_{\mathsf{e}\Sigma^1_1}}\, \mathsf{M}_1 \models \Gamma.$$

(ii) If $x \in Q$ with $\deg(x) = m{+}2$ and $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^*_{\mathsf{e}\Sigma^1_{m+1}}(\mathsf{T}|\mathsf{S})$, then

$$\overset{*}{\mathsf{T}}_x|\mathsf{S} \,\frac{\gamma}{+}\, \Gamma \;\Rightarrow\; \overset{*}{\mathsf{T}}_{x(\beta)}|\mathsf{S} \,\frac{<E(\gamma)}{*}\, \Gamma, \text{ for some } \beta < \gamma.$$

If $\deg(x) > 1$, then property (ii) immediately lead to a derivation that is simpler in that $|(x(\beta), E(\gamma))| < |(x,\gamma)|$. If $\deg(x) = 1$ and $o(x) = 1$, so $x = y{+}1$ for some y, then property (i) allowed us to use the I.H., essentially as $|(x(\beta), g_{x^h(\alpha)})| < |(x, \alpha{+}1)|$. Finally, if $\deg(x) = 1$ and $o(x) = \gamma$ (so $\breve{\mathsf{T}}_x$ iff $(\forall \beta \lhd o(x))\mathsf{p}_1(\breve{\mathsf{T}}_{x[\beta]})$), then the I.H. allowed us to eliminate cuts with $\mathcal{L}^*$-instances of $\mathsf{p}_1(\breve{\mathsf{T}}_{x[\beta]})$, since for $\beta < \gamma$, $|(x[\beta]{+}1, g_{x^h(\alpha)})| < |(x, \alpha{+}1)|$. It is important, though, to note that the negation of such an instance is Π^1_1, and thus, after using inversion, we are left with an arithmetical sequent.

Next, we review the proof of the stronger result $(\forall x \in Q)\mathsf{Bd}_{n+1}(x)$, as the very same strategy was applied in the first part to show $(\forall x \in Q)\mathsf{Prv}_{n+1}(x)$. We emphasize that we do not use induction along $(Q, <)$. This is tempting, since reduction property (ii) yields $\forall\alpha \mathsf{Bd}_0(x(\alpha)) \to \mathsf{Bd}_0(x)$, a result that readily extends, for each n, to

$$(*) \qquad \forall\alpha \mathsf{Bd}_{n+1}(x(\alpha)) \to \mathsf{Bd}_{n+1}(x).$$

However, $x(\alpha)$ is not $\rightsquigarrow^*$-smaller than x, only $x(\alpha) < x$ w.r.t. $(Q, <)$. And even if working in some unspecified meta-theory, it makes no sense to assume the well-foundedness of $(Q, <)$ to prove a result whose purpose is mainly to obtain the proof-theoretic ordinal of T_x and thus its consistency, when $|\mathsf{T}_x|$ is much smaller than the ordertype of $(Q, <)$.

To show $\mathsf{Bd}_m(x)$, we fixed N_0 so that $x \in Q_{N_0+1-m}$, and proved by meta-induction on $n < N_0$, that $(\forall x \in Q_{n+1})\mathsf{Bd}_{N_0-n}(x)$. The meta-induction step was shown by a side-induction along $\rightsquigarrow^*$. Letting $n := N_0 - m$, we have got $\mathsf{Bd}_m(x)$.

We start by reviewing the side-induction step for names x with $\deg(x) = 1$. As the direct proof, using induction along $\rightsquigarrow^*$ instead of $|(x, \gamma)|$, we saw that for such names, $\mathsf{Bd}_0(x)$ follows, if $\mathsf{Bd}_0(x[\beta])$ for all $\beta < o(x)$ (if $o(x)$ is a limit, then also an end-piece suffices). Again, this was readily extended in the following way: for each $x \in Q$ with $\deg(x) = 1 \wedge o(x) = \delta_0+\gamma$, and each $(1, v) \in Q$,

(a) $(\forall \alpha < \gamma)\mathsf{Bd}_{n+1}(x[\delta_0+\alpha]) \to \mathsf{Bd}_{n+1}(x)$,

(b) $\mathsf{Bd}_{n+1}(1, v) \to \forall \alpha \mathsf{Bd}_{n+1}(1+\alpha, v)$.

Both claims were shown simultaneously by induction on n. (b) was shown by transfinite induction on α. Only the limit case required some thought, and was obtained using (a) doing the characteristic two-step approximation (see the proof of Lemma III.7.13, or page 66 for a more verbose discussion).

An easy induction on n now yielded $\mathsf{Bd}_{n+2}(q_1)$: using (b), we saw that $\mathsf{Bd}_{n+1}(1, y)$ implies $\forall \alpha \mathsf{Bd}_{n+1}((1+\alpha, y)+1)$, i.e., $\forall \alpha \mathsf{Bd}_{n+1}((1+\alpha, y)(\alpha))$, so $\mathsf{Bd}_{n+1}(1, y+1)$ by $(*)$. This helps with the side-induction step for names x with $\deg(x) > 1$ (see below).

Now we return to the proof of $(\forall x \in Q_{n+1})\mathsf{Bd}_{N_0-n}(x)$. If $n = 0$, then the claim reads $(\forall x \in Q_1)\mathsf{Bd}_{N_0}(x)$. For $q_0 \in Q_1$, the claim holds trivially, and $\mathsf{Bd}_{N_0}(q_1)$ is by the above observation, so $\mathsf{Bd}_{N_0}(1+\alpha, q_0)$ is by (b). Hence $(\forall x \in Q_1)\mathsf{Bd}_{N_0}(x)$. Concerning the meta-induction step, we assumed that $n+1 < N_0$ and $(\forall x \in Q_{n+1})\mathsf{Bd}_{N_0-n}(x)$, and showed by induction along $\rightsquigarrow^*$, that $(\forall x \in Q_{n+2})\mathsf{Bd}_{N_0-n-1}(x)$.

If $\deg(x) = 1$, then by the I.H., $(\forall \alpha \lhd \gamma)\mathsf{Bd}_{N_0-n-1}(x[\alpha])$, thus $\mathsf{Bd}_{N_0-n-1}(x)$ by (a). And if $\deg(x) > 1$, we decomposed x into $x =_{NF} (1, y) \circ z$ with $\deg(y) > 0$. Since $z \rightsquigarrow^* x$, we have $\mathsf{Bd}_{N_0-n-1}(z)$, and $\mathsf{Bd}_{N_0-n}(y)$ is by the meta-I.H. (since $(1, y) \in Q_{n+2}$ and $\deg(y) > 0$, thus $y \in Q^*_{n+1}$), which together with $\mathsf{Bd}_{N_0-n-1}(q_1)$ yields $\mathsf{Bd}_{N_0-n-1}(1, y)$ (note that $q_1 = (1, q_0)$), so $\mathsf{Bd}_{n+1}(x)$ follows.

To be clear: if e.g. $x =_{NF} (1, y) \circ z$, and $\overset{*}{\mathsf{T}}_x \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$, then we do not reduce this derivation to $\overset{*}{\mathsf{T}}_{x(\beta)} \mathrel{\big|\!\frac{<E(\gamma)}{*}} \Gamma$ for some $\beta < \gamma$. Instead, we regard $\overset{*}{\mathsf{T}}_x \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$ as $(\mathsf{Op}^+_y \circ \mathsf{p}_1)(\overset{*}{\mathsf{T}}_z) \mathrel{\big|\!\frac{<\gamma}{+}} \Gamma$. Now we have $\mathsf{Bd}_2(y)$ by the meta-I.H., which together with $\mathsf{Bd}_1(q_1)$ yields $\mathsf{Bd}_1(1, y+1)$, and $\mathsf{Bd}_1(x)$ follows from $\mathsf{Bd}_1(z)$.

Finally, we revisit the proof of $\mathsf{T}^\epsilon \vdash (\forall x \in Q^*_{N_0})\mathsf{Prv}_{n+1}(x)$ presented in the first part. Since T^ϵ comprises no (transfinite) induction at all, we constantly employed Theorem I.4.2, to have a substitute for transfinite induction. We attempted to approximate "$\mathsf{T}^\epsilon \vdash \mathsf{Prv}_n(x)$" by $\mathsf{prv}_n(x) := \forall X \mathsf{Prv}(x)_n{\restriction}X$, stating that $\mathsf{Prv}_n(x)$ holds in all ω-models of T^ϵ. Then, by the theorem and the form of the formula

$\mathsf{Prv}_n(u)$, $(\forall x \in Q)\mathsf{Prv}_n(x)$ is a consequence of the following variant of the induction step: $\forall x[(\forall y \rightsquigarrow^* x)\mathsf{prv}_n(y) \to \mathsf{Prv}_n(x)]$.

Since by the form of $\mathsf{Prv}_n(x)$ we have only a substitute for transfinite induction along $\rightsquigarrow^*$, we had no choice but to show $(\forall x \in Q^*)\mathsf{Prv}_1(x)$ using provable functions of higher types. To make this claim at least plausible, we recall that, similarly to $\forall\alpha\mathsf{Bd}_{n+1}(x(\alpha)) \to \mathsf{Bd}_{n+1}(x)$, we have got $\forall\alpha\mathsf{prv}_{n+1}(x(\alpha)) \to \mathsf{Prv}_{n+1}(x)$ (now working in T^ϵ). Thus, an obvious idea to obtain $\mathsf{Prv}_1(x)$ from $(\forall y \rightsquigarrow^* x)\mathsf{prv}_1(y)$ is to show $\forall\alpha\mathsf{Prv}_1(x(\alpha))$ using induction on α and the assumptions $\mathsf{Prv}_1(x(0))$ and $\mathsf{prv}_1(x(0))$, and to leave the problem of how to obtain $\mathsf{Prv}_1(x(0))$ for later. If say $\deg(x) = m+2$ with $o(x) \in \mathsf{Lim}(\Omega)$ and $x =_{NF} y \circ_m z$ with $z = L((1, y_1) \circ_1 \ldots \circ_1 (1, y_{m-1}))$, then $x(\alpha+1) = (y(0) \circ y(\alpha)) \circ_m z$ (cf. Definition III.4.6). In this situation, $\mathsf{prv}_{m+1}(y(\alpha))$ and $\mathsf{Prv}_1(z)$ yield $\mathsf{Prv}_1(y(\alpha) \circ_m z)$, that is, $\mathsf{Prv}_1(x(\alpha))$, as an easy induction on m reveals. We consider this evidence that a detour via provable functionals is required.

When we were computing bounds, reduction property (i) allowed us to step form $\mathsf{Bd}_0(x)$ to $\mathsf{Bd}_0(x+1)$. In the context of provable functions, property (i) corresponds in some sense to

(i)' $\mathsf{ACA}_0 \vdash \breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \wedge \mathsf{Wo}_\lhd(\alpha) \to \mathsf{Wo}_\lhd(g_{x^h}(\alpha))$ (cf. Lemma III.7.9).

Since $\breve{\mathsf{T}}_{x+1}$ implies that above any Y there is a X with $Y \dot{\in} X$ and $\breve{\mathsf{T}}_{x(\alpha)}\restriction X$, having $\mathsf{prv}_0(x)$ at hand, and thus $\mathsf{Prv}_0(x)\restriction X$, allowed us to conclude $\mathsf{TI}_\lhd(Y, g_{x^h}(\alpha))$, so $\mathsf{Wo}_\lhd(g_{x^h}(\alpha))$: the relativized assumption $\mathsf{TI}_\lhd(\mathcal{C}_x\restriction X, \alpha)$ in $\mathsf{Prv}_0(x)\restriction X$ is immediate by $\mathsf{Wo}_\lhd(\alpha)$, as $\mathcal{C}_x\restriction X$ is a set.

Reduction property (ii) corresponds to

(ii)' $\mathsf{T}^\epsilon \vdash \deg(x) > 1 \wedge \breve{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\{\alpha : \breve{\mathsf{T}}_{x(\alpha)}\})$ (cf. Lemma III.6.13).

Using (ii)', $\breve{\mathsf{T}}_x$ and $\mathsf{TI}_\lhd(\mathcal{C}_x, \alpha)$ entail $\breve{\mathsf{T}}_{x(\alpha)}$, and also that for each Y, there is an X with $Y \dot{\in} X$ and $\breve{\mathsf{T}}_{x(\alpha)}\restriction X$. With $\mathsf{prv}_0(x(\alpha))$ at hand, we obtained $\mathsf{Wo}_\lhd(g_{x^h[\alpha]}(\beta))$ similarly as above. Hence, $\forall\alpha\mathsf{prv}_0(x(\alpha)) \to \mathsf{Prv}_0(x)$ is an immediate consequence of (ii)', which was readily lifted to $\forall\alpha\mathsf{prv}_{n+1}(x(\alpha)) \to \mathsf{Prv}_{n+1}(x)$.

This time, we showed by meta-induction on $n < N_0$, that for all $n < N_0$, $\mathsf{ACA}_0 \vdash (\forall x \in Q^*_{n+1})\mathsf{Prv}_{N_0-n}(x)$. Again, the meta-induction step was shown by transfinite induction along $\rightsquigarrow^*$; for names with $\deg(x) = 1$, we first employed Lemma III.7.10, which states that ACA_0 proves the following.

1) $x \in Q_{N_0} \wedge \breve{\mathsf{T}}_{x+1} \wedge \mathsf{prv}_0(x) \to \mathsf{Prog}_\lhd(\mathcal{C}_{x+1})$.

2) $x \in Q^*_{N_0} \wedge \deg(x) = 1 \wedge o(x) = \gamma \wedge (\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\alpha]) \wedge \breve{\mathsf{T}}_x \to \mathsf{Prog}_\lhd(\mathcal{C}_x)$.

We saw that $\deg(x) = 1$ implies $(\forall\alpha \lhd \gamma)\mathsf{prv}_0(x[\alpha]) \to \mathsf{Prv}_0(x)$: if $x = y+1$, this is by 1) and the observation that $\mathsf{TI}_\lhd(\mathcal{C}_{y+1}, \alpha)$ yields $\alpha \in \mathcal{C}_{y+1}$, that is, $\mathsf{Wo}_\lhd(g_{y^h+1}(\alpha))$, then indeed $\mathsf{prv}_0(y) \to \mathsf{Prv}_0(y{+}1)$; if $o(x) \in \mathsf{Lim}(\Omega)$, the claim follows along the same lines using 2).

Again, $(\forall \alpha \lhd \gamma)\mathsf{prv}_0(x[\alpha]) \to \mathsf{Prv}_0(x)$ extended to the following (see Lemma III.7.13): for each n and each $x \in Q$ with $\deg(x) = 1 \wedge o(x) = \delta_0{+}\gamma$, and each $(1, v) \in Q$,

(a)' $(\forall \alpha \lhd \gamma)\mathsf{prv}_{n+1}(x[\delta_0{+}\alpha]) \to \mathsf{Prv}_{n+1}(x)$,

(b)' $\mathsf{Prv}_{n+1}(1, v) \to \forall \alpha \mathsf{Prv}_{n+1}(1{+}\alpha, v)$.

Then, an induction on n was used to show that $\mathsf{ACA}_0 \vdash \mathsf{Prv}_{n+1}(q_1)$, which helped to settle the side-induction step for names with $\deg(x) > 1$.

Now $\mathsf{ACA}_0 \vdash (\forall x \in Q^*_{n+1})\mathsf{Prv}_{N_0-n}(x)$ was shown by meta-induction on $n < N_0$ and side induction along $\rightsquigarrow^*$, analogously to $(\forall x \in Q_{n+1})\mathsf{Bd}_m(x)$.

Chapter VII

Appendix

1 Proof of the Representation Theorem

We start by recalling relevant notions and by proving an auxiliary lemma.

For an $\mathsf{L}_2(\mathsf{P})$-formula $\varphi(u)$, $\varphi(U,u)\{\breve{\mathsf{T}}\restriction X\}$ is obtained from $\varphi(U,u)$ by replacing each occurrence of $\mathsf{P}(\mathcal{X})$ in φ by the formula $\breve{\mathsf{T}}\restriction\mathcal{X}$. Note that thus e.g. $\varphi(U,u)\{\breve{\mathsf{T}}\restriction U\}$ and $\varphi(U,u)\{\breve{\mathsf{T}}\restriction X\}$ are the same formula. We write $\varphi_{\breve{\mathsf{T}}\restriction U}(U,u)$ for $\varphi(U,u)\{\breve{\mathsf{T}}\restriction U\}$. If $\varphi(u)$ represents an operation, then Op_u^φ is the operation that maps $\breve{\mathsf{T}}$ to $\varphi_{\breve{\mathsf{T}}\restriction U}(u)$. Further, recall Lemma I.2.19, which is tacitly used in the sequel.

A set is called transitive, if $\mathsf{trans}\restriction X$, where $\mathsf{trans} = \forall X, x \exists Y[Y = (X)_x]$. Hence, if $\mathsf{trans}\restriction X$, then $\forall x, y[(X)_{x,y}\dot{\in}X]$, and also $\forall x,y,z[(X)_{x,y,z}\dot{\in}X]$. The next auxiliary lemma states that given trans, then for a Σ^1_1-formula $B(U) := \exists Y A(Y,U)$, we have $X \models (\vartheta\{B(U)\})$ is equivalent to $X \models (\vartheta\{B'(U)\})$, where $B'(U) := U\dot{\in}X \wedge B(U)\restriction X$. Essentially, this holds since the range of $\exists Y$ is restricted anyway after substituting $\exists Y A(Y,\mathcal{X})$ for $\mathsf{P}(\mathcal{X})$, and since each set variable W that occurs in some $\mathsf{P}(\mathcal{X})$ and remains free after substituting is replaced by $(X)_w$. The transitivity of X is required to obtain $\mathcal{X}\dot{\in}X$ from $W\dot{\in}X$, in case $\mathcal{X}$ is of the form $(W)_{\vec{s}}$.

Lemma A.1.1. *Assume that $A(V,U)$ is an arithmetical L_2-formula, that ϑ is an $\mathsf{L}_2(\mathsf{P}^+)$ formula, and $B(U) := \exists Y A(Y,U)$ and $B'(U) := U\dot{\in}X \wedge (\exists Y\dot{\in}X)A(Y,U)$. Then T^ϵ proves:*

$$\mathsf{trans}\restriction X \to [X \models (\vartheta\{B(U)\}) \leftrightarrow X \models (\vartheta\{B'(U)\})].$$

Proof We prove the claim by induction on the build-up of ϑ. As it is assumed that the set variables occurring in $B'(U)$ and ϑ are disjoint, X will not occur in ϑ. Assume that X is transitive. Exemplarily we treat the following cases.

(i) $\vartheta = \mathsf{P}(\mathcal{X})$ and $\mathcal{X} = (W)_s$. We have to verify that $X \models \exists Y A(Y,(W)_s)$ iff $X \models \exists y A((X)_y,(W)_s) \wedge (W)_s\dot{\in}X$, i.e., $\exists y A((X)_y,(X)_{w,s})$ iff $\exists y A((X)_y,(X)_{w,s})\wedge$

$(X)_{w,s}\dot{\in}X$, where w is a fresh variable. As $\mathsf{trans}{\restriction}X$ implies $(X)_{w,s}\dot{\in}X$, this is the case.

(ii) $\vartheta = \forall V\psi(V,W)$, $\mathsf{FV}_1(\vartheta) = \{W\}$ and W does not occur free in $A(Y,U)$. For $B(U) := \exists Y A(Y,U)$, $B'(U) := U\dot{\in}X \wedge (\exists Y\dot{\in}X)A(Y,U)$, the I.H. yields $X \models \psi(V,W)\{B(U)\}$ iff $X \models \psi(V,W)\{B'(U)\}$, that is, $\psi((X)_v,(X)_w)\{B(U)\}{\restriction}X$ iff $\psi((X)_v,(X)_w)\{B'(U)\}{\restriction}X$, where v,w are fresh variables. Quantifying v on both sides, and observing (using Lemma I.2.18) that for C either B or B',

$$\forall v(\psi((X)_v,(X)_w)\{C(U)\}{\restriction}X) = (\forall V\psi(V,(X)_w)\{C(U)\}){\restriction}X = \\ X \models \forall V\psi(V,W)\{C(U)\} = X \models \vartheta\{C(U)\}$$

yields the claim.

□

Below, we restate the Representation Theorem (Theorem I.2.26) in a slightly more explicit form. We let $\mathsf{Wog}_{\rightsquigarrow,\prec}(u) := \mathsf{Wo}_{\prec}(u) \wedge \mathsf{good}(\rightsquigarrow,\prec)$. Recall that $\mathsf{good}(\rightsquigarrow,\prec)$ is an arithmetical sentence that asserts that $\prec$ is the transitive closure of $\rightsquigarrow$ (cf. Definition I.2.24).

Theorem A.1.2. *Suppose that $\vartheta(u)$ is an open $\mathcal{L}(\mathsf{P}^+)$-sentence that strongly implies p_1, and that $\prec,\rightsquigarrow,f(u,v)$ are primitive recursive. Then, there is an $\mathcal{L}(\mathsf{P}^+)$-formula $\psi(U,u)$ which is Σ^1_1 with exactly the displayed variables free,*

$$\varphi(u) := \varphi^{f,\prec,\rightsquigarrow,\vartheta}(u) := 0 \prec u \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(u) \ \wedge\ (\forall x \rightsquigarrow u)\vartheta(f(x,u))\{\psi(X,x)\}$$

strongly implies p_1, and T^ϵ proves that if $0 \prec u$ and $\mathsf{Wog}_{\rightsquigarrow,\prec}(u)$, then

$$(*) \qquad \mathsf{Op}^{\varphi}_u(\breve{\mathsf{T}}) \leftrightarrow (\forall x \rightsquigarrow u)(\mathsf{Op}^{\vartheta}_{f(x,u)}(\widehat{\mathsf{Op}}^{\varphi}_x(\breve{\mathsf{T}}))).$$

Throughout this section, which is devote to the proof of this theorem, $\vartheta(u)$ and $\varphi(u)$ are as assumed in the theorem. As $\vartheta(u)$ strongly implies p_1, T^ϵ proves that $\vartheta(f(x,u))\{\psi(X,x)\}$ implies $\varphi_{\mathsf{p}_1}\{\psi(X,x)\}$, a fact that we use tacitly in the sequel. Further, we let $\alpha,\beta,\ldots$ range over $\mathsf{field}(\prec)$ and γ over limits w.r.t. $\prec$. We think of $\prec$ as a well-founded ordering with $\mathsf{Wo}_{\prec}(\delta)$ for each $\delta \in \mathsf{field}(\prec)$, and $\mathsf{good}(\rightsquigarrow,\prec)$. If the context suggests that we consider an element $\alpha \prec \delta$, then we write $\alpha{+}1$ for $\mathsf{min}\{\beta : \alpha \prec \beta \preceq \delta\}$ (recall that $\mathsf{Wo}_{\prec}(\delta)$ implies that $\{\beta : \alpha \prec \beta \preceq \delta\}$ is well-ordered).

Firstly, we comment on the form of the formula $\varphi(u)$ introduced in the above theorem. By definition, $\varphi(u)$ contains P only positively. Further, for each L_2-formula $A(U)$, $\vartheta(u)\{\psi(X,\alpha)\}\{A(U)\}$ is $\vartheta(u)\{\psi_{A(U)}(X,\alpha)\}$ (Lemma I.2.22 (ii)). Next, we argue that $\varphi(u)$ represents an operation. It remains to show that $\mathsf{T}^\epsilon \vdash \varphi_\top(u) \to \mathsf{pair} \wedge \mathsf{trans}$. We work informally in T^ϵ and assume $\varphi_\top(u)$. If $0 \not\prec u \vee \neg\mathsf{Wog}_{\rightsquigarrow,\prec}(\prec)$, then $\mathsf{pair} \wedge \mathsf{trans}$ follows trivially, and if $0 \prec u \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(\prec)$, then there is an α with $\alpha \rightsquigarrow u$, hence $\vartheta(f(\alpha,u))\{\psi_\top(X,\alpha)\}$. Since each L_2-formula trivially implies $\top$, Lemma I.2.19 yields $\vartheta_\top(f(\alpha,u))$. Further, as ϑ represents an operation, for each y, $\vartheta_\top(y)$ implies $\mathsf{pair} \wedge \mathsf{trans}$. Therefore, also $\varphi_\top(u)$ implies $\mathsf{pair} \wedge \mathsf{trans}$.

Secondly, we explain the idea leading to this form of φ. Assume for the moment, that $\varphi(u)$ is of the form specified in the formulation of the theorem, that $0 \prec \delta$ and $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$, and that $(*)$ holds for each $\alpha \prec \delta$. Further, assume that the $\mathsf{L}_2(\mathsf{P})$-formula $\psi(U,u)$ is such that that for $D_1(U,\alpha) := (\alpha = 0 \wedge \check{\mathsf{T}}{\restriction}U)$ and $D_2(U,\alpha) := (0 \prec \alpha \wedge \mathsf{Op}^\varphi_\alpha(\check{\mathsf{T}}){\restriction}U)$,

$$(**) \qquad (\forall\alpha \prec \delta)[\psi(X,\alpha)\{\check{\mathsf{T}}{\restriction}U\} \leftrightarrow D_1(X,\alpha) \vee D_2(X,\alpha)].$$

By choice of φ we have that

$$\varphi_{\check{\mathsf{T}}\restriction U}(\delta) \leftrightarrow (\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{\psi_{\check{\mathsf{T}}\restriction U}(X,\alpha)\}.$$

Further, by the assumption $(**)$, we have that $\forall X[\psi_{\check{\mathsf{T}}\restriction U}(X,0) \leftrightarrow D_1(X,0)]$, and if $0 \prec \alpha$, then $\forall X[\psi_{\check{\mathsf{T}}\restriction U}(X,\alpha) \leftrightarrow D_2(X,\alpha)]$. Therefore, $\varphi_{\check{\mathsf{T}}\restriction U}(\delta)$ is equivalent to

$$(\forall\alpha \rightsquigarrow \delta)[(0 = \alpha \wedge \vartheta(f(\alpha,\delta))\{D_1(X,\alpha)\}) \vee (0 \prec \alpha \wedge \vartheta(f(\alpha,\delta))\{D_2(X,\alpha)\})],$$

which by definition of D_1 and D_2 (recall: $\vartheta(z)\{\check{\mathsf{T}}'{\restriction}U\}) = \mathsf{Op}^\vartheta_z(\check{\mathsf{T}}')$) is equivalent to

$$(\forall\alpha \rightsquigarrow \delta)[(0 = \alpha \wedge \mathsf{Op}^\vartheta_{f(\alpha,\delta)}(\check{\mathsf{T}})) \vee (0 \prec \alpha \wedge \mathsf{Op}^\vartheta_{f(\alpha,\delta)}(\mathsf{Op}^\varphi_\alpha(\check{\mathsf{T}})))],$$

which is equivalent to $(\forall\alpha \rightsquigarrow \delta)(\mathsf{Op}^\vartheta_{f(\alpha,\delta)}(\widehat{\mathsf{Op}}^\varphi_\alpha(\check{\mathsf{T}})))$. Hence $(*)$ holds also for δ.

Before we can give the definition of $\psi(X,\delta)$, we need some terminology.

Definition A.1.3. *If $Z = (X)_y$, we say that y is an X-index of Z. Further, $Z \dot\in_Y X$ abbreviates $(\exists y \in Y)[Z = (X)_y]$, i.e. that "$Z$ has an X-index in Y".*

We refer to Y as a set of X-indices, if we think of each $y \in Y$ as an index of the set $(X)_y$. Now we consider the following hierarchy of X-indices. Its definition contains a notational ambiguity that we resolve here: $\vartheta(n)\{U \dot\in_{(Y)_\beta} X\}$ (and also $\vartheta(n)\{V \dot\in_{(Y)_\beta} X\}$) is obtained by replacing each expression of the form $\mathsf{P}(\mathcal{X})$ in $\vartheta(n)$ by $\mathcal{X} \dot\in_{(Y)_\beta} X$.

Definition A.1.4. $\mathsf{Hier}_\mathsf{T}(Y,X,\delta) := \chi_{\check{\mathsf{T}}\restriction U}(Y,X,\delta)$*, where $\chi(Y,X,\delta)$ is the conjunction of the formulas listed below:*

(i) $0 \prec \delta \to [(Y)_0 = \{y : \mathsf{P}((X)_y)\}]$,

(ii) $(\forall\alpha \prec \delta)[0 \prec \alpha \to (Y)_\alpha = \bigcap_{\beta \rightsquigarrow \alpha}\{y : \vartheta(f(\beta,\alpha))\{U \dot{\in}_{(Y)_\beta} X\}\restriction(X)_y\}]$.

Note that only (i) is an $\mathsf{L}_2(\mathsf{P})$-formula, as in (ii), $\mathsf{P}(\mathcal{X})$ is replaced by $\mathcal{X}\dot{\in}_{(Y)_\beta}X$. To see more clearly what the above formula claims, assume that $0 \prec \delta$ and $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$ and $\mathsf{Hier}_\mathsf{T}(Y,X,\delta)$, and additionally, that X is transitive. Then, for each $\alpha \prec \delta$, $(Y)_\alpha$ is a set of X-indices. Some properties of the hierarchy Y are discussed below.

(i) $(Y)_0$ contains the X-indices of models of T, i.e. $(Y)_0 = \{y : \breve{\mathsf{T}}\restriction(X)_y\}$.

(ii) By definition, $y \in (Y)_1$ iff $\vartheta(f(0,1))\{U\dot{\in}_{(Y)_0}X\}\restriction(X)_y$. As by (i), $U\dot{\in}_{(Y)_0}X$ implies $\breve{\mathsf{T}}\restriction U$, we also have that $y \in (Y)_1$ implies $\vartheta(f(0,1))\{\breve{\mathsf{T}}\restriction U\}\restriction(X)_y$, and since ϑ strongly implies p_1, also $\varphi_{\mathsf{p}_1}\{\breve{\mathsf{T}}\restriction U\}\restriction(X)_y$. So $(X)_y$ is a model of $\mathsf{p}_1(\mathsf{T})$ and thus transitive. Since X is transitive, $\breve{\mathsf{T}}\restriction U \wedge U\dot{\in}(X)_y$ iff $U\dot{\in}_{(Y)_0}X \wedge U\dot{\in}(X)_y$. Applying Lemma A.1.1 to the transitive set $(X)_y$ and the formulas $A(V,U) := \breve{\mathsf{T}}\restriction U$ and $B(U) := \exists Z(\breve{\mathsf{T}}\restriction U)$, and then $A(V,U) := (U\dot{\in}_{(Y)_0}X)$ and $B(U) := \exists Z(U\dot{\in}_{(Y)_0}X)$, ($\exists Z$ is just a dummy quantifier, so e.g. $U\dot{\in}(X)_y \wedge (\exists Z\dot{\in}(X)_y)(\breve{\mathsf{T}}\restriction U)$ is $\breve{\mathsf{T}}\restriction U \wedge U\dot{\in}(X)_y$) yields

$$y \in (Y)_1 \leftrightarrow \vartheta(f(0,1))\{U\dot{\in}_{(Y)_0}X\}\restriction(X)_y \leftrightarrow \vartheta(f(0,1))\{\breve{\mathsf{T}}\restriction U\}\restriction(X)_y.$$

Therefore, $(Y)_1$ contains the X-indices of models of $\mathsf{Op}^{\vartheta}_{f(0,1)}(\mathsf{T})$.

(iii) By the same reasoning, assuming that $(Y)_\alpha$ contains the X-indices of models of $\mathsf{Op}_\alpha(\mathsf{T})$, $(Y)_{\alpha+1}$ contains the X-indices of models of $\mathsf{Op}^{\vartheta}_{f(\alpha,\alpha+1)}(\mathsf{Op}_\alpha(\mathsf{T}))$, hence of $\mathsf{Op}_{\alpha+1}(\mathsf{T})$.

Lemma A.1.5. T^ϵ *proves that if* $0 \prec \delta$ *and* $\mathsf{Hier}_\mathsf{T}(Y,X,\delta)$, *then*

(i) $U\dot{\in}_{(Y)_0}X \to \breve{\mathsf{T}}\restriction U$,

(ii) $0 \prec \alpha \prec \delta \wedge U\dot{\in}_{(Y)_\alpha}X \to (\forall\beta \rightsquigarrow \alpha)\vartheta(f(\beta,\alpha))\{V\dot{\in}_{(Y)_\beta}X\}\restriction U$,

(iii) $\alpha \prec \delta \wedge U\dot{\in}_{(Y)_\alpha}X \to \mathsf{trans}\restriction U$.

Proof Assume that $0 \prec \delta$ and $\mathsf{Hier}_\mathsf{T}(Y,X,\delta)$. (i) If $U\dot{\in}_{(Y)_0}X$, then there is a y with $U = (X)_y \wedge y \in (Y)_0$. As $y \in (Y)_0$ iff $\breve{\mathsf{T}}\restriction(X)_y$, $\breve{\mathsf{T}}\restriction U$ follows. (ii) If $U\dot{\in}_{(Y)_\alpha}X$, then there is a y with $U = (X)_y \wedge y \in (Y)_\alpha$. Therefore, $\mathsf{Hier}_\mathsf{T}(Y,X,\delta)$ implies $(\forall\beta \rightsquigarrow \alpha)\vartheta(f(\beta,\alpha))\{V\dot{\in}_{(Y)_\beta}X\}\restriction(X)_y$, hence the claim. (iii) If $\alpha = 0$, then (i) yields $\mathsf{trans}\restriction U$, and if $0 \prec \alpha \prec \delta \wedge U\dot{\in}_{(Y)_\alpha}X$, then (ii) and the fact that $\vartheta(u)$ represents an operation entails $\mathsf{trans}\restriction U$. □

We will see that X is model of $\mathsf{Op}_\delta(\mathsf{T})$, if there is a "good" Y, so that $\mathsf{Hier}_\mathsf{T}(Y,X,\delta)$. Next, we are looking for conditions that make such a Y "good". For each $\alpha \prec \delta$, $(Y)_\alpha$ should contain enough X-indices. It is plausible that for each $V\dot{\in}X$, there should be a W so that $W\dot{\in}_{(Y)_\alpha}X$ and $V\dot{\in}W$. This condition follows from $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction X$, which unwinds to $(\forall V\exists W(V\dot{\in}W \wedge W\dot{\in}_{(Y)_\alpha}X) \wedge \mathsf{pair} \wedge \mathsf{trans})\restriction X$.

If $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction X$ holds for some stage $\alpha \prec \delta$, then also for all stages $\beta \prec \alpha$. This is what the next lemma claims if we choose $Z := X$.

Lemma A.1.6. *The following is provable in* ACA_0*:*

$$\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta) \wedge \mathsf{Hier}_{\mathsf{T}}(Y,X,\delta) \wedge \beta \prec \alpha \prec \delta \wedge \varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction Z \to \varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\beta}X\}\restriction Z.$$

Proof Assume $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$ and $\mathsf{Hier}_{\mathsf{T}}(Y,X,\delta)$. We fix some β with $\beta+1 \prec \delta$ and show by induction on α, that

$$\beta \prec \alpha \prec \delta \wedge \varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction Z \to \varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\beta}X\}\restriction Z.$$

Suppose that $\beta \prec \alpha \prec \delta$ and that the claim holds for all $\alpha' \prec \alpha$. As $\mathsf{good}(\rightsquigarrow,\prec)$, there is an $\alpha_0 \prec \alpha$ with $\beta \preceq \alpha_0 \rightsquigarrow \alpha$. We assume $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction Z$ and aim for $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_{\alpha_0}}X\}\restriction Z$. The claim then follows by the I.H. To show $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_{\alpha_0}}X\}\restriction Z$, we fix some $V\dot{\in}Z$. $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\restriction Z$ implies that there is a $W'\dot{\in}Z$ with $W'\dot{\in}_{(Y)_\alpha}X$ and $V\dot{\in}W'$. $\alpha_0 \rightsquigarrow \alpha$ and $W'\dot{\in}_{(Y)_\alpha}X$ imply $\vartheta(f(\alpha_0,\alpha))\{U\dot{\in}_{(Y)_{\alpha_0}}X\}\restriction W'$ by Lemma A.1.5, and since ϑ strongly implies φ_{p_1}, also $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_{\alpha_0}}X\}\restriction W'$. Hence there is a $W\dot{\in}W'$ so that $W\dot{\in}_{(Y)_{\alpha_0}}X$ and $V\dot{\in}W$. This shows that $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_{\alpha_0}}X\}\restriction Z$. $\square$

Corollary A.1.7. ACA_0 *proves:*

$$\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta) \wedge \mathsf{Hier}_{\mathsf{T}}(Y,X,\delta) \wedge (\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{U\dot{\in}_{(Y)_\alpha}X\}\restriction X \to$$
$$(\forall\beta \prec \delta)\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\beta}X\}\restriction X.$$

Proof If $\beta \prec \delta$, there is an α with $\beta \preceq \alpha \rightsquigarrow \delta$. As ϑ strongly implies φ_{p_1}, $\vartheta(f(\alpha,\delta))\{U\dot{\in}_{(Y)_\alpha}X\}$ implies $\mathsf{p}_1\{U\dot{\in}_{(Y)_\alpha}X\}$, and $\mathsf{p}_1\{U\dot{\in}_{(Y)_\beta}X\}$ is by Lemma A.1.6. $\square$

In fact, if $\mathsf{Hier}_{\mathsf{T}}(Y,X,\delta)$, then $(\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{U\dot{\in}_{(Y)_\alpha}X\}\restriction X$ is a sufficient condition to ensure $\mathsf{Op}_\delta(\breve{\mathsf{T}})\restriction X$. Hence, we let $\psi(X,\delta)$ claim that there exists a hierarchy meeting the above condition. For technical reasons discussed below, we add another condition.

Definition A.1.8. $\psi(X,\delta) := \exists Y\psi'(X,Y,\delta)$, *where* $\psi'(X,Y,\delta)$ *is the conjunction of the following formulas.*

(i) $0 = \delta \to \mathsf{P}(X)$,

(ii) $0 \prec \delta \to \chi(Y,X,\delta) \wedge \varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_0}X\}\restriction X$,

(iii) $0 \prec \delta \to (\forall \alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{U \dot{\in}_{(Y)_\alpha} X\}\restriction X$.

Clearly, ψ is Σ^1_1. Again, only (i) and $\chi(Y, X, \delta)$ contain the relation symbol P. Further, $\psi(X, 0)$ iff $\mathsf{P}(X)$, so $\psi_{\breve{\mathsf{T}}\restriction U}(X, 0)$ iff $\breve{\mathsf{T}}\restriction X$. And if $0 \prec \delta$, then $\psi_{\breve{\mathsf{T}}\restriction U}(X, \delta)$ implies that there is a Y such that $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$ and $\varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y)_0} X\}\restriction X$, and also $(\forall \alpha \rightsquigarrow \delta)\vartheta_{\breve{\mathsf{T}}\restriction U}(f(\alpha,\delta))\{U \dot{\in}_{(Y)_\alpha} X\}\restriction X$. As discussed above, $U \dot{\in}_{(Y)_0} X$ implies $\breve{\mathsf{T}}\restriction U$, therefore $0 \prec \delta$ and $\psi_{\breve{\mathsf{T}}\restriction U}(X, \delta)$ imply $\mathsf{p}_1(\breve{\mathsf{T}})\restriction X$.

Lemma A.1.9. T^ϵ *proves the following.*

(i) $\psi'_{\breve{\mathsf{T}}\restriction U}(X, Y, \delta) \to \mathsf{Hier}_{\mathsf{T}}(X, Y, \delta)$,

(ii) $\psi_{\breve{\mathsf{T}}\restriction U}(X, 0) \leftrightarrow \breve{\mathsf{T}}\restriction X$,

(iii) $0 \prec \delta \wedge \psi_{\breve{\mathsf{T}}\restriction U}(X, \delta) \to \varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y)_0} X\}\restriction X$,

(iv) $0 \prec \delta \wedge \psi_{\breve{\mathsf{T}}\restriction U}(X, \delta) \to \mathsf{p}_1(\breve{\mathsf{T}})\restriction X$.

Also note that by (ii) and (iv), $\psi_{\breve{\mathsf{T}}\restriction U}(X, \delta)$ yields $(\mathsf{ACA})\restriction X$. By Corollary A.1.7, we also have the following

Lemma A.1.10. $\mathsf{ACA}_0 \vdash \mathsf{Wog}_{\rightsquigarrow,\prec}(\prec) \wedge \psi_{\breve{\mathsf{T}}\restriction U}(X, \delta) \to (\forall \alpha \prec \delta)\varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y)_\alpha} X\}\restriction X$.

In view of Corollary A.1.7, the second conjunct of (ii) seems superfluous. We added it so that the second claim of Lemma A.1.12 is provable in T^ϵ.

Now that we have discussed all the ingredients, we restate the definition of $\varphi(u)$.

Definition A.1.11.

(i) $\varphi'(u) := (\forall \alpha \rightsquigarrow u)\vartheta(f(\alpha, u))\{\psi(X, \alpha)\}$, *and*

(ii) $\varphi(u) := 0 \prec u \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(u) \wedge \varphi'(u)$.

Lemma A.1.12. $\mathsf{T}^\epsilon \vdash \varphi_{\breve{\mathsf{T}}\restriction U}(1) \to \mathsf{p}_1(\breve{\mathsf{T}})$ *and* $\mathsf{T}^\epsilon \vdash 1 \prec \delta \wedge \varphi_{\breve{\mathsf{T}}\restriction U}(\delta) \to \mathsf{p}_1^2(\breve{\mathsf{T}})$.

Proof Assume $\varphi_{\breve{\mathsf{T}}\restriction U}(1)$. By definition, $\varphi_{\breve{\mathsf{T}}\restriction U}(1)$ implies $\vartheta(f(0,1))\{\psi_{\breve{\mathsf{T}}\restriction U}(X, 0)\}$. As ϑ strongly implies p_1, we obtain $\varphi_{\mathsf{p}_1}\{\psi_{\breve{\mathsf{T}}\restriction U}(X, 0)\}$. Since $\psi_{\breve{\mathsf{T}}\restriction U}(X, 0)$ iff $\breve{\mathsf{T}}\restriction X$, $\mathsf{p}_1(\breve{\mathsf{T}})$ follows. For the second claim, assume $1 \prec \delta \wedge \varphi(\delta)$. For an α with $1 \preceq \alpha \rightsquigarrow \delta$, we obtain $\varphi_{\mathsf{p}_1}\{\psi_{\breve{\mathsf{T}}\restriction U}(X, \alpha)\}$ as above. By Lemma A.1.9, $\psi_{\breve{\mathsf{T}}\restriction U}(X, \alpha)$ implies $\mathsf{p}_1(\breve{\mathsf{T}})\restriction X$, hence $\varphi_{\mathsf{p}_1}\{\psi_{\breve{\mathsf{T}}\restriction U}(X, \alpha)\}$ implies $\varphi_{\mathsf{p}_1}\{\mathsf{p}_1(\breve{\mathsf{T}})\restriction X\}$, which is $\mathsf{p}_1^2(\breve{\mathsf{T}})$. ☐

As discussed below, claim $(*)$ of Theorem A.1.2 is easily obtained, once we have the following lemma, which we prove at the end of this section.

Lemma A.1.13. $\mathsf{p}_1(\mathsf{ACA}_0) \vdash 0 \prec \delta \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(\delta) \to (\psi_{\breve{\mathsf{T}}\restriction U}(X, \delta) \leftrightarrow \varphi_{\breve{\mathsf{T}}\restriction U}(\delta)\restriction X)$.

We argue why this lemma implies that T^ϵ proves that $0 \prec \delta \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$ imply $(*)$ of Theorem A.1.2. So assume $0 \prec \delta \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$. If $\delta = 1$, $(*)$ holds as $\varphi_{\breve{\mathsf{T}}\restriction U}(1)$ iff $\varphi'_{\breve{\mathsf{T}}\restriction U}(1)$ iff $\vartheta(f(0,1))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,0)\}$ iff $\vartheta(f(0,1))\{\breve{\mathsf{T}}\restriction U\}$ iff $\mathsf{Op}^{\vartheta}_{f(0,1)}(\breve{\mathsf{T}})$. And for $1 \prec \delta$, $\varphi_{\breve{\mathsf{T}}\restriction U}(\delta)$ implies $\mathsf{p}^2_1(\breve{\mathsf{T}})$. Thus Lemma A.1.13 is at hand. Hence, $\varphi_{\breve{\mathsf{T}}\restriction U}(\delta)$ iff $\varphi'_{\breve{\mathsf{T}}\restriction U}(\delta)$ iff $(\forall \alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,\alpha)\}$ iff

$$(\forall \alpha \rightsquigarrow \delta)[(0 = \alpha \wedge \vartheta(f(\alpha,\delta))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,0)\}) \vee (0 \prec \alpha \wedge \vartheta(f(\alpha,\delta))\{\psi_{\breve{\mathsf{T}}\restriction U}(X,\alpha)\})],$$

iff

$$(\forall \alpha \rightsquigarrow \delta)[(0 = \alpha \wedge \vartheta(f(\alpha,\delta))\{\breve{\mathsf{T}}\restriction U\}) \vee (0 \prec \alpha \wedge \vartheta(f(\alpha,\delta))\{\varphi_{\breve{\mathsf{T}}\restriction U}(\delta)\restriction X)\})].$$

As $\varphi_{\breve{\mathsf{T}}\restriction U}(\delta)$ is $\mathsf{Op}^{\varphi}_{\delta}(\breve{\mathsf{T}})$ and $\vartheta(f(\alpha,\delta))\{\varphi_{\breve{\mathsf{T}}\restriction U}(\delta)\restriction X)\}$ is $\mathsf{Op}^{\vartheta}_{f(\alpha,\delta)}(\mathsf{Op}^{\varphi}_{\delta}(\breve{\mathsf{T}}))$, the displayed formula is just another way of writing $(\forall \alpha \rightsquigarrow \delta)(\mathsf{Op}^{\vartheta}_{f(\alpha,\delta)}(\widehat{\mathsf{Op}}^{\varphi}_{\alpha}(\breve{\mathsf{T}})))$.

Before we prove the above lemma, we need some further notions and auxiliary results dealing with indices of sets.

Definition A.1.14. $U \dot{\subseteq} V := \forall x \exists y[(U)_x = (V)_y]$.

Hence if $X \dot{\in} U$ and $U \dot{\subseteq} V$, then also $X \dot{\in} V$.

Suppose that Y is a set of X-indices. For some set W with $W \dot{\in}_Y X$ we may also have $W \dot{\in} X'$, so W has also X'-indices. Below, we define $Y_{X'/X}$ so that it contains the X'-indices of the sets with X-indices in Y. Further, if Y is a hierarchy of X-indices, then we define $Y_{[X'/X]}$ so that for each α and $Y' := (Y)_\alpha$ and $(Y_{[X'/X]})_\alpha = Y'_{X'/X}$.

Definition A.1.15.

(i) $Y_{X'/X} := \{y' : (\exists y \in Y)[(X')_{y'} = (X)_y]\}$, *and*

(ii) $Y_{[X'/X]} := \{\langle y', \alpha\rangle : (\exists y \in (Y)_\alpha)[(X')_{y'} = (X)_y]\}$.

Lemma A.1.16. *The following is provable in* T^ϵ.

(i) $y' \in Y_{X'/X}$ *iff* $(X')_{y'} \dot{\in}_Y X$,

(ii) $W \dot{\in}_{Y_{X'/X}} X'$ *iff* $W \dot{\in} X' \wedge W \dot{\in}_Y X$,

(iii) $(\forall W \dot{\in} X')(W \dot{\in}_Y X)$ *iff* $(\forall W \dot{\in} X')(W \dot{\in}_{(Y_{X'/X})_{X/X'}} X)$,

(iv) *if* $X' \dot{\subseteq} X''$, *then* $Y_{X'/X} = (Y_{X''/X})_{X'/X''}$ *and* $Y_{[X'/X]} = (Y_{[X''/X]})_{[X'/X'']}$.

Proof (i) $y' \in Y_{X'/X}$ iff there is a $y \in Y$ with $(X')_{y'} = (X)_y$ iff $(X')_{y'} \dot{\in}_Y X$. (ii) If $W \dot{\in}_{Y_{X'/X}} X'$, then clearly $W \dot{\in} X'$, and since $W = (X)_{y'}$ for some $y' \in Y_{X'/X}$, (i) yields that $W = (X')_{y'} \dot{\in}_Y X$. Conversely, if $W \dot{\in} X'$ and $W \dot{\in}_Y X$, then there is a y' with $W = (X')_{y'} \dot{\in}_Y X$, so $y' \in Y_{X'/X}$, and thus $W \dot{\in}_{Y_{X'/X}} X'$. (iii) Assume that $(\forall W \dot{\in} X')(W \dot{\in}_Y X)$. Hence if $W \dot{\in} X'$, then also $W \dot{\in}_Y X$, and so by (ii), $W \dot{\in}_{Y_{X'/X}} X'$. As also $W \dot{\in} X$, using (ii) again yields $W \dot{\in}_{(Y_{X'/X})_{X/X'}} X$. For the other direction, observe that by (ii), $W \dot{\in}_{(Y_{X'/X})_{X/X'}} X$ yields $W \dot{\in}_{Y_{X'/X}} X'$, so $W \dot{\in} X'$ and $W \dot{\in}_Y X$. (iv) Let $y' \in Y_{X'/X}$. Then $(X')_{y'} \dot{\in}_Y X$, and there is a $y \in Y$ with $(X')_{y'} = (X)_y$. As $X' \dot{\subseteq} X''$, there is a y'' with $(X'')_{y''} = (X)_y$, thus $(X'')_{y''} \dot{\in}_Y X$, that is, $y'' \in Y_{X''/X}$. As $(X')_{y'} = (X'')_{y''}$, $(X')_{y'} \dot{\in}_{Y_{X''/X}} X''$, which says that $y' \in (Y_{X''/X})_{X'/X''}$. Conversely, if $y' \in (Y_{X''/X})_{X'/X''}$, then $(X')_{y'} \dot{\in}_{Y_{X''/X}} X''$. Hence, there is $y'' \in Y_{X''/X}$ so that $(X')_{y'} = (X'')_{y''}$. But $y'' \in Y_{X''/X}$ says that $(X'')_{y''} \dot{\in}_Y X$, hence also $(X')_{y'} \dot{\in}_Y X$, that is, $y' \in Y_{X'/X}$. The second claim follows easily from the first. □

Lemma A.1.17. T^ϵ *proves the following: if Z is transitive, $Z \dot{\subseteq} X'$ and $Z \dot{\subseteq} X$, then $\vartheta\{U \dot{\in}_Y X\}{\restriction}Z \leftrightarrow \vartheta\{U \dot{\in}_{Y_{X'/X}} X'\}{\restriction}Z$.*

Proof Assume that Z is transitive, and that $Z \dot{\subseteq} X'$, $Z \dot{\subseteq} X$. If $W \dot{\in} Z$, then $W \dot{\in} X$ iff $W \dot{\in} X'$, and so $W \dot{\in}_Y X$ iff $W \dot{\in}_{Y_{X'/X}} X'$ by Lemma A.1.16 (ii). As ϑ has no free set variables and since Z is transitive, we have, using Lemma A.1.1, that $\vartheta\{U \dot{\in}_Y X\}{\restriction}Z$ iff $\vartheta\{U \dot{\in} Z \wedge U \dot{\in}_Y X\}{\restriction}Z$ iff $\vartheta\{U \dot{\in} Z \wedge U \dot{\in}_{Y_{X'/X}} X'\}{\restriction}Z$ iff $\vartheta\{U \dot{\in}_{Y_{X'/X}} X'\}{\restriction}Z$. □

As a consequence, we obtain the next lemma.

Lemma A.1.18. T^ϵ *proves the following: if $X' \dot{\in} X$, X' and X are transitive and $Y' := Y_{X'/X}$, then $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta) \to \mathsf{Hier}_{\mathsf{T}}(Y', X', \delta)$.*

Proof If $\delta = 0$ there is nothing to show. So assume that $\alpha \prec \delta$, $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$ and $Y' := Y_{[X'/X]}$. We have to show that

(i) $y' \in (Y')_0$ iff $\breve{\mathsf{T}}{\restriction}(X')_{y'}$, and

(ii) if $0 \prec \alpha \prec \delta$, then $y' \in (Y')_\alpha \leftrightarrow (\forall \beta \rightsquigarrow \alpha)\vartheta(f(\beta, \alpha))\{U \dot{\in}_{(Y')_\beta} X'\}{\restriction}(X')_{y'}$.

By definition of Y', $y' \in (Y')_\alpha$ iff $(X')_{y'} \dot{\in}_{(Y)_\alpha} X$. If $\alpha = 0$, then $y \in (Y)_0$ iff $\breve{\mathsf{T}}{\restriction}(X)_y$. Hence $y' \in (Y')_0$ iff $(X')_{y'} \dot{\in}_{(Y)_0} X$ iff $\breve{\mathsf{T}}{\restriction}(X')_{y'} \wedge (X')_{y'} \dot{\in} X$ iff $\breve{\mathsf{T}}{\restriction}(X')_{y'}$, since $X' \dot{\subseteq} X$ ($X' \dot{\in} X$ and $\mathsf{trans}{\restriction}X$), thus (i). And if $0 \prec \alpha \prec \delta$, then $y \in (Y)_\alpha$ iff $A(X, Y){\restriction}(X')_{y'}$ for $A(X, Y) := (\forall \beta \rightsquigarrow \alpha)\vartheta(f(\beta, \alpha))\{U \dot{\in}_{(Y)_\beta} X\}$. Hence $y \in (Y)_\alpha$ iff $(X')_{y'} \dot{\in}_{(Y)_\alpha} X$ iff $A(X, Y){\restriction}(X')_{y'} \wedge (X')_{y'} \dot{\in} X$ iff $A(X', Y'){\restriction}(X')_{y'}$ since $X' \dot{\subseteq} X$ and by Lemma A.1.17 (as $Z := (X')_{y'} \dot{\in}_{(Y)_\alpha} X$, Z is transitive, and since X and X' are transitive, we have $Z \dot{\subseteq} X'$ and $Z \dot{\subseteq} X$, hence the Lemma applies). This shows (ii). □

For the above lemma to apply, we require that X' and X are transitive. In the situations where we use the this lemma, these sets are transitive for the following reason: $\psi'(W, Y, \delta)$ as well as $W \dot{\in}_{Y_\alpha} X \wedge \alpha \prec \delta \wedge \mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$ imply the transitivity of W (cf. Lemma A.1.5 and A.1.9).

Lemma A.1.19. ACA_0 *proves the following: if* $Y' := Y_{[X'/X]}$*, then*

$$\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta) \wedge \psi'_{\check{\mathsf{T}}\restriction U}(X, Y, \delta) \wedge \alpha \prec \delta \wedge X' \dot{\in}_{(Y)_\alpha} X \to \psi'_{\check{\mathsf{T}}\restriction U}(X', Y', \alpha).$$

Proof Assume $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$, $\psi'_{\check{\mathsf{T}}\restriction U}(X, Y, \delta)$, $\alpha \prec \delta$ and $X' \dot{\in}_{(Y)_\alpha} X$, and let $Y' := Y_{[X'/X]}$. Note, that $\psi'_{\check{\mathsf{T}}\restriction U}(X, Y, \delta)$ entails $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$, so also $\mathsf{Hier}_{\mathsf{T}}(Y, X, \alpha)$ for $\alpha \prec \delta$. If $\alpha = 0$, then $X' \dot{\in}_{(Y)_0} X$ yields $\check{\mathsf{T}}\restriction X'$, thus $\psi'_{\check{\mathsf{T}}\restriction U}(X', Y', 0)$. And if $0 \prec \alpha$, then $\psi'_{\check{\mathsf{T}}\restriction U}(X', Y', \alpha)$ holds if

(i) $\mathsf{Hier}_{\mathsf{T}}(Y', X', \alpha) \wedge \varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y')_0} X'\}\restriction X'$, and

(ii) $(\forall \beta \rightsquigarrow \alpha)\vartheta(f(\beta, \alpha))\{U \dot{\in}_{(Y')_\beta} X'\}\restriction X'$.

$\mathsf{Hier}_{\mathsf{T}}(Y', X', \alpha)$ follows form $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$ by Lemma A.1.18, as $X' \dot{\in}_{(Y)_\alpha} X$ and thus transitive. The second conjunct of (i) follows from (ii) (which we show next) by Corollary A.1.7. (ii) follows in two steps. By Lemma A.1.5, $X' \dot{\in}_{(Y)_\alpha} X$ yields $(\forall \beta \rightsquigarrow \alpha)\vartheta(f(\beta, \alpha))\{U \dot{\in}_{(Y)_\beta} X\}\restriction X'$ by Lemma A.1.5, and now (ii) is by Lemma A.1.17. □

We already know that if $X' \dot{\in} X$, X and X' are transitive, and $\mathsf{Hier}_{\mathsf{T}}(Y, X, \delta)$, then also $\mathsf{Hier}_{\mathsf{T}}(Y', X', \delta)$ for $Y' := Y_{[X'/X]}$. Next, we show that under suitable assumptions, $(Y')_{\prec\beta} \dot{\in} X$ for each $\beta \prec \delta$.

Lemma A.1.20. *Consider the formula*

$$B(X, X', Y, \delta) := \psi'_{\check{\mathsf{T}}\restriction U}(X, Y, \delta) \wedge X' \dot{\in} X \to (\forall \alpha \prec \delta)((Y_{[X'/X]})_{\prec\alpha} \dot{\in} X).$$

Then, $\mathsf{p}_1(\mathsf{ACA}_0) \vdash \mathsf{Wog}_{\rightsquigarrow,\prec}(\delta_0) \to B(X, X', Y, \delta_0)$.

Proof Let $\mathcal{C}_{\delta_0} := \{\delta : \delta \preceq \delta_0 \to \forall X, X', Y\, B(X, X', Y, \delta)\}$. We show that ACA_0 proves $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta_0) \to \mathsf{Prog}_{\prec}(\mathcal{C}_{\delta_0})$, thus $\mathsf{T}^\epsilon \vdash (\mathsf{ACA})\restriction W \wedge \mathsf{Wog}^W_{\rightsquigarrow,\prec}(\delta_0) \to \mathsf{Prog}_{\prec}(\mathcal{C}_{\delta_0}\restriction W)$. Then, working in $\mathsf{p}_1(\mathsf{ACA}_0)$, we assume $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta_0)$, fix sets X, Y, X' with $X' \dot{\in} X$, $\alpha \prec \delta_0$, and pick a set W with $X, Y \dot{\in} W$ and $(\mathsf{ACA})\restriction W$. With $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta_0)$, we also have $\mathsf{Wog}^W_{\rightsquigarrow,\prec}(\delta_0)$. Therefore, we have $\mathsf{Prog}_{\prec}(\mathcal{C}_{\delta_0}\restriction W)$, and thus $\delta_0 \in \mathcal{C}_{\delta_0}\restriction W$. Hence $B(X, X', Y, \delta_0)$, which implies $(Y_{[X'/X]})_{\prec\alpha} \dot{\in} X$.

Now we work informally in ACA_0 and show $\mathsf{Prog}_{\prec}(\mathcal{C}_{\delta_0})$. Trivially, $0 \in \mathcal{C}_{\delta_0}$. Next, assume that $0 \prec \delta \preceq \delta_0$ and $(\forall \beta \prec \delta)(\beta \in \mathcal{C}_{\delta_0})$. To show that $\delta \in \mathcal{C}_{\delta_0}$, let $\alpha \prec \delta$ and X, X', Y so that $X' \dot{\in} X$ and $\psi'_{\mathsf{T}\restriction U}(X, Y, \delta)$. Set $Y' := Y_{[X'/X]}$ and aim for

$(Y')_{\prec\alpha}\dot{\in}X$. $\psi'_{\mathsf{T}\upharpoonright U}(X,Y,\delta)$ entails $\mathsf{p}_1(\check{\mathsf{T}})\upharpoonright X$ by Lemma A.1.9. Further $\psi'_{\mathsf{T}\upharpoonright U}(X,Y,\delta)$ and Lemma A.1.10 yield $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(Y)_\alpha}X\}\upharpoonright X$. This states that there is a set $X''\dot{\in}_{(Y)_\alpha}X$ with $X'\dot{\in}X''$. So X'' is transitive and $X'\dot{\subseteq}X''$. Let $Y'':=Y_{[X''/X]}$. By Lemma A.1.19, $\psi'_{\mathsf{T}\upharpoonright U}(X'',Y'',\alpha)$. As $Y'=Y''_{[X'/X'']}$ by Lemma A.1.16 (iv), the I.H. (i.e. $B(X'',X',Y'',\alpha)$) implies that for each $\beta\prec\alpha$, $(Y')_{\prec\beta}\dot{\in}X''$.

By Lemma A.1.18, $\mathsf{Hier}_{\mathsf{T}}(Y',X',\delta)$. So if $\alpha=\beta{+}1$, then $(Y')_{\prec\alpha}=(Y')_{\prec\beta}\cup(Y')_\beta$, which is arithmetical in $(Y')_{\prec\beta}$ and X', since by definition of $\mathsf{Hier}_{\mathsf{T}}(Y',X',\delta)$,

$$(Y')_\beta=\bigcap_{\xi\rightsquigarrow\beta}\{y:\vartheta(f(\xi,\beta))\{U\dot{\in}_{(Y')_\xi}X'\}\upharpoonright(X')_y\}.$$

Thus $(Y')_{\prec\alpha}\dot{\in}X$. If $\alpha=:\gamma$ is a limit, then we have $\mathsf{Hier}_{\mathsf{T}}((Y')_{\prec\beta},X',\beta)$ for each $\beta\prec\gamma$, and the relevant part of the hierarchy Y' is unique: if also $\mathsf{Hier}_{\mathsf{T}}(Z,X',\beta)$, then $(Z)_{\prec\beta}=(Y')_{\prec\beta}$. Further, for each $\beta\prec\gamma$, $(Y')_{\prec\beta}\dot{\in}X''$ is by I.H. Therefore,

$$(Y')_{\prec\gamma}=\{\langle y,\beta\rangle:\beta\prec\gamma\wedge\exists w[\mathsf{Hier}_{\mathsf{T}}((X'')_w,X',\beta)\wedge\langle y,\beta\rangle\in(X'')_w]\}.$$

So $(Y')_{\prec\gamma}$ is arithmetical in X' and X'', and so $(Y')_{\prec\gamma}\dot{\in}X$. Hence $\delta\in\mathcal{C}_{\delta_0}$. □

For the remainder of this subsection, we fix a hierarchy $H^X_{\prec\delta}$ w.r.t. X and $\check{\mathsf{T}}$. The hierarchy $H^X_{\prec\delta}$ is such that for $\xi\not\prec\delta$, $(H^X_{\prec\delta})_\xi=\emptyset$, and otherwise,

(i) $(H^X_{\prec\delta})_0=\{y:\check{\mathsf{T}}\upharpoonright(X)_y\}$, and for $0\prec\alpha\prec\delta$,

(ii) $(H^X_{\prec\delta})_\alpha=\{y:(\exists Y\dot{\in}X)\psi'_{\check{\mathsf{T}}\upharpoonright U}((X)_y,Y,\alpha)\}$.

Note that is $H^X_{\prec\delta}$ arithmetical in X. We also point out that

$$U\dot{\in}_{(H^X_{\prec\delta})_\alpha}X\ \leftrightarrow\ U\dot{\in}X\wedge(\exists Y\dot{\in}X)\psi'_{\check{\mathsf{T}}\upharpoonright U}(U,Y,\alpha),$$

as $U\dot{\in}_{(H^X_{\prec\delta})_\alpha}X$ iff $\exists y[U=(X)_y\wedge(\exists Y\dot{\in}X)\psi'_{\check{\mathsf{T}}\upharpoonright U}(U,Y,\alpha)]$.

Now we prove the left-to-right direction of Lemma A.1.13.

Lemma A.1.21. $\mathsf{p}_1(\mathsf{ACA}_0)\vdash 0\prec\delta\wedge\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)\wedge\varphi(\delta)\upharpoonright X\rightarrow\psi'_{\check{\mathsf{T}}\upharpoonright U}(X,H^X_{\prec\delta},\delta)$.

Proof Assume that $0\prec\delta$ and $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$ and $\varphi(\delta)\upharpoonright X$. We have to check that

(i) $\mathsf{Hier}_{\mathsf{T}}(H^X_{\prec\delta},X,\delta)$,

(ii) $\varphi_{\mathsf{p}_1}\{U\dot{\in}_{(H^X_{\prec\delta})_0}X\}\upharpoonright X$, and

(iii) $(\forall\alpha\rightsquigarrow\delta)\vartheta(f(\alpha,\delta))\{U\dot{\in}_{(H^X_{\prec\delta})_\alpha}X\}\upharpoonright X$.

First, we show (iii). $\varphi_{\check{\mathsf{T}}\restriction U}(\delta)\restriction X$ yields $(\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{\psi(X,\alpha)\}\restriction X$ by definition. By Lemma A.1.1, we obtain $(\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{A(U)\}\restriction X$, for $A(U) := U\dot\in X \wedge (\exists Y\dot\in X)\psi'_{\check{\mathsf{T}}\restriction U}(U,Y,\alpha)$. Further, by definition of $H^X_{\prec\delta}$, we have $A(U)$ iff $U\dot\in_{(H^X_{\prec\delta})_\alpha}X$, thus $(\forall\alpha \rightsquigarrow \delta)\vartheta(f(\alpha,\delta))\{U\dot\in_{(H^X_{\prec\delta})_\alpha}X\}\restriction X$, hence (iii).

Now Corollary A.1.7 yields $(\forall\alpha \prec \delta)\varphi_{\mathsf{p}_1}\{U\dot\in_{(H^X_{\prec\delta})_\alpha}X\}\restriction X$. This implies (ii), and further states that $(\forall\alpha \prec \delta)\forall V\exists W[V\dot\in W \wedge W\dot\in_{(H^X_{\prec\delta})_\alpha}X]\restriction X$, which implies

$$(**) \qquad (\forall\alpha \prec \delta)(\forall V\dot\in X)(\exists W\dot\in X)(\exists Y\dot\in X)[V\dot\in W \wedge \psi'_{\check{\mathsf{T}}\restriction U}(W,Y,\alpha)].$$

It remains to show $\mathsf{Hier}_{\mathsf{T}}(H^X_{\prec\delta},X,\delta)$. We let $Y := H^X_{\prec\delta}$, and verify that Y is formed according to Definition A.1.4. For $\alpha = 0$ this is evident, and for $0 \prec \alpha \prec \delta$, we show by transfinite induction on α that

$$(*) \qquad y \in (Y)_\alpha \quad \text{iff} \quad (\forall\beta \rightsquigarrow \alpha)\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y)_\beta}X\}\restriction(X)_y.$$

Firstly, note that by Definition A.1.4, if $(*)$ holds for all $\alpha' \prec \alpha$, then $\mathsf{Hier}_{\mathsf{T}}(Y,X,\alpha)$. Secondly, by definition of Y, $y \in (Y)_\alpha$ iff $\psi_{\check{\mathsf{T}}\restriction U}((X)_y,\alpha)\restriction X$ iff there is a $Z\dot\in X$, so that

(i) $\mathsf{Hier}_{\mathsf{T}}(Z,(X)_y,\alpha)$ and

(ii) $\varphi_{\mathsf{p}_1}\{U\dot\in_{(Z)_0}(X)_y\}\restriction(X)_y$, and

(iii) $(\forall\beta \rightsquigarrow \alpha)\vartheta(f(\beta,\alpha))\{U\dot\in_{(Z)_\beta}(X)_y\}\restriction(X)_y$.

Now we show the two directions of $(*)$, assuming that $(*)$ already holds for all $\alpha' \prec \alpha$. Keep in mind that $(X)_y$ and X are transitive, so $(X)_y \dot\subseteq X$.

Left-to-right: By I.H., $\mathsf{Hier}_{\mathsf{T}}(Y,X,\alpha)$. For $Y' := Y_{[(X)_y/X]}$, Lemma A.1.18 implies $\mathsf{Hier}_{\mathsf{T}}(Y',(X)_y,\alpha)$. Hence, for the witness Z in (i), $(Y')_{\prec\alpha} = (Z)_{\prec\alpha}$. Fix a $\beta \rightsquigarrow \alpha$. By (iii) we have $\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y')_\beta}(X)_y\}\restriction(X)_y$. For $Y'' := Y'_{[X/(X)_y]}$, Lemma A.1.17 yields $\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y'')_\beta}X\}\restriction(X)_y$. Hence, $\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y)_\beta}X\}\restriction(X)_y$ by Lemma A.1.16 (iii). This concludes the proof of the left-to-right direction of $(*)$.

Right-to-left: Assume that $(\forall\beta \rightsquigarrow \alpha)\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y)_\beta}X\}\restriction(X)_y$. Again by I.H. $\mathsf{Hier}_{\mathsf{T}}(Y,X,\alpha)$, and Lemma A.1.18 yields $\mathsf{Hier}_{\mathsf{T}}((Y_0)_{\prec\alpha},(X)_y,\alpha)$ for $Y_0 := Y_{[(X)_y/X]}$. The right hand side of $(*)$ and Lemma A.1.17 yield $\vartheta(f(\beta,\alpha))\{U\dot\in_{(Y_0)_\beta}(X)_y\}\restriction(X)_y$ for each $\beta \rightsquigarrow \alpha$. Now Corollary A.1.7 yields $(\forall\beta \prec \alpha)\varphi_{\mathsf{p}_1}\{U\dot\in_{(Y_0)_\beta}(X)_y\}\restriction(X)_y$, hence $\psi'_{\check{\mathsf{T}}\restriction U}((X)_y,(Y_0)_{\prec\alpha},\alpha)$. It remains to show that $(Y_0)_{\prec\alpha}\dot\in X$. First, we pick a δ' with $\alpha \preceq \delta' \rightsquigarrow \delta$. By $(**)$, there are $W,Y'\dot\in X$ with $(X)_y\dot\in W$ so that $\psi'_{\check{\mathsf{T}}\restriction U}(W,Y',\delta')$, notably $\mathsf{Hier}_{\check{\mathsf{T}}}(Y',W,\delta')$. By Lemma A.1.18, $\mathsf{Hier}_{\check{\mathsf{T}}}((Y'')_{\prec\delta'},(X)_y,\delta')$, for $Y'' := Y'_{[(X)_y/W]}$ and $Y''\dot\in X$ as X satisfies arithmetical comprehension. Hence $(Y_0)_{\prec\alpha} = (Y'')_{\prec\alpha}\dot\in X$. This concludes the verification of $(*)$. $\square$

The other direction of Lemma A.1.13 is proved next.

Lemma A.1.22. $\mathsf{p}_1(\mathsf{ACA}_0) \vdash 0 \prec \delta \wedge \mathsf{Wog}_{\rightsquigarrow,\prec}(\delta) \wedge \psi_{\check{\mathsf{T}}\restriction U}(X,\delta) \to \varphi_{\check{\mathsf{T}}\restriction U}(\delta)\restriction X.$

Proof We work informally in $\mathsf{p}_1(\mathsf{ACA}_0)$. Assume $0 \prec \delta$, $\mathsf{Wog}_{\rightsquigarrow,\prec}(\delta)$ and that there is a Y with $\psi'_{\check{\mathsf{T}}\restriction U}(X,Y,\delta)$. Fix $\alpha \rightsquigarrow \delta$. Now $\psi'_{\check{\mathsf{T}}\restriction U}(X,Y,\delta)$ yields $\vartheta(f(\alpha,\delta))\{U \dot{\in}_{(Y)_\alpha} X\}\restriction X$. If we can show that for each X', $X' \dot{\in}_{(Y)_\alpha} X$ implies $\psi_{\check{\mathsf{T}}\restriction U}(X',\alpha)\restriction X$, then we also have $\vartheta(f(\alpha,\delta))\{\psi_{\check{\mathsf{T}}\restriction U}(U,\alpha)\}\restriction X$, which then gives $\varphi'_{\check{\mathsf{T}}\restriction U}(\delta)\restriction X$. Indeed, by Lemma A.1.19, $\psi'_{\check{\mathsf{T}}\restriction U}(X,Y,\delta)$ and $X' \dot{\in}_{(Y)_\alpha} X$ yield $\psi'_{\check{\mathsf{T}}\restriction U}(X',(Y')_{\prec\alpha},\alpha)$ for $Y' := Y_{X'/X}$, and by Lemma A.1.20, we conclude that $(Y')_{\prec\alpha} \dot{\in} X$. $\square$

Finally, we show that $\varphi(\delta)$ strongly implies p_1.

Lemma A.1.23. *φ strongly implies φ_{p_1}.*

Proof Let $A(U)$ be a Σ^1_1-formula of L_2. By definition, $\varphi_{A(U)}(u)$, implies $0 \prec u$ and $(\forall \alpha \rightsquigarrow u)\vartheta(f(\alpha,\delta))\{\psi_{A(U)}(X,\alpha)\}$. As $0 \prec u$, there is an $\alpha \rightsquigarrow u$ for which we have $\vartheta(f(\alpha,u))\{\psi_{A(U)}(X,\alpha)\}$, therefore also $\varphi_{\mathsf{p}_1}\{\psi_{A(U)}(X,\alpha)\}$, as ϑ strongly implies p_1. If $\alpha = 0$, then $\psi_{A(U)}(X,\alpha)$ iff $A(X)$ by definition of ψ, and thus $\varphi_{\mathsf{p}_1}\{A(U)\}$. And if $0 \prec \alpha$, then the definition of ψ provides a Y, so that $\varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y)_0} X\}\restriction X$ and $(Y)_0 = \{y : A((X)_y)\}$, i.e. $U \dot{\in}_{(Y)_0} X$ iff $A(U) \wedge U \dot{\in} X$. Hence $\varphi_{\mathsf{p}_1}\{U \dot{\in}_{(Y)_0} X\}\restriction X$ also yields $\varphi_{\mathsf{p}_1}\{A(U)\}\restriction X$. Summing up, $\varphi_{\mathsf{p}_1}\{\psi_{A(U)}(X,\alpha)\}$ implies $\varphi_{\mathsf{p}_1}\{\varphi_{\mathsf{p}_1}\{A(U)\}\restriction X\}$, that is, $\mathsf{p}_1(\varphi_{\mathsf{p}_1}\{A(U)\})$. As $\varphi_{\mathsf{p}_1}\{A(U)\}$ is Π^1_2, $\varphi_{\mathsf{p}_1}\{A(U)\}$ follows by Lemma I.2.12. $\square$

2 The theory Π^0_1-CA_0^-

For the definition of the theory Π^0_1-CA_0^- it proves convenient to have bounded quantifiers at hand. Therefore, we extent the language L_2 as follows: if $A(u)$ is an L_2-formula and t a number term with $u \notin \mathsf{FV}_0(t)$, then $(\forall x < t)A(x)$ and $(\exists x < t)A(x)$ are L_2-formulas, too. This gives rise to the following classes of formulas. The set of Δ^0_0-formulas (also called Π^0_0 or Σ^0_0) of L_2 contains all literals of L_2 and is closed under conjunction, disjunction and bounded number quantification. Further, A is Π^0_{n+1} $[\Sigma^0_{n+1}]$, if A is Π^0_n, $[\Sigma^0_n]$ or of the form $\forall x B(x)$ $[\exists x B(x)]$ with B Σ^0_n $[\Pi^0_n]$.

Definition A.2.1. *Think of e as an index of a unary partial recursive function, and let $\{e\}(x) = y$ be a Σ^0_1-formula of L_2 expressing that y is the value of this function with index e applied to the number x. Then, we set*

(i) $\forall$-$\mathsf{CL} := \forall X \forall e \exists Y[Y = \{x : \forall y, z(\{e\}(\langle x, y\rangle) = z \to z \in X)\}]$.

(ii) $\exists^b$-$\mathsf{CL} := \forall X \forall e \forall a \exists Y[Y = \{x : (\exists y < a)\forall z(\{e\}(\langle x, y\rangle) = z \to z \in X)\}]$.

Definition A.2.2. Π^0_1-$\mathsf{CA}_0^- := \mathsf{T}^\epsilon + \bigwedge\{\forall X \mathsf{IND}(X), \mathsf{pair}, \mathsf{trans}, \forall\text{-}\mathsf{CL}, \exists^b\text{-}\mathsf{CL}\}$.

Lemma A.2.3. *For each Π^0_1-formula A that contains all its set variables only positively, Π^0_1-CA_0^- proves that $\{x : A(\vec{X}, x, \vec{y})\}$ is a set.*

Proof N is a set. Let e so that $\forall x\{e\}(x)\uparrow$ (i.e. $\forall x, y(\{e\}(x) \neq z)$). Then, we have that $\mathsf{N} = \{x : \forall y, z(\{e\}(\langle x, y\rangle) = z \to z \in X)\}$. $\emptyset$ is a set: pick an a that does not code a pair and let e so that $\forall x, y\{e\}(\langle x, y\rangle) = a)$. Then, $\emptyset = \{x : \forall y, z(\{e\}(\langle x, y\rangle) = z \to z \in \mathsf{N}+\mathsf{N})\}$. For each primitive recursive $R(\vec{x})$, there is an e, so that for all y, $\{e\}(\langle\langle\vec{x}\rangle, y\rangle)\uparrow$ iff $R(\vec{x})$. Hence $R = \{\langle\vec{x}\rangle : \forall y\{e\}(\langle\langle\vec{x}\rangle, y\rangle)\uparrow\} = \{\langle\vec{x}\rangle : \forall y, z[\{e\}(\langle\langle\vec{x}\rangle, y\rangle) = z \to z \in \emptyset)\}$. The same holds true for $\sim R(\vec{x})$. Now one shows by induction on the build-up of $A(\vec{u})$, that $\{\langle\vec{x}\rangle : A(\vec{x})\}$ is a set. The claim then follows. If $A = t(\vec{u}) \in U$, then let e so that $\forall \vec{x}, y(\{e\}(\langle\langle\vec{x}\rangle, y\rangle) = t(\vec{x}))$. Then $\{\langle\vec{x}\rangle : t(\vec{x}) \in U\} = \{\langle\vec{x}\rangle : \forall y, z(\{e\}(\langle\langle\vec{x}\rangle, y\rangle) = z \to z \in U)\}$. The case $A = R(t_1(\vec{x}), \dots, t_n(\vec{x}))$ is handled similarly. To handle conjunction, bounded and unbounded universal quantification, use pair and $\forall$-CL. Exemplarily, we consider the case $A(U, u) = \forall z B(U, u, z)$. By I.H. we know that $Y := \{\langle x, y\rangle : B(U, x, y)\}$ is a set. Then, for and e so that $\forall x, y(\{e\}(\langle x, y\rangle) = \langle x, y\rangle)$, $\{\langle x\rangle : A(U, x)\} = \{\langle x\rangle : \forall y, z(\{e\}(\langle x, y\rangle) = z \to z \in Y)\}$. For disjunction and bounded existential quantification, use pair and $\exists^b$-CL. $\square$

Remark A.2.4. *The theory $\mathsf{p}_1((\Pi^0_1\text{-}\mathsf{CA})^-)$ is not stronger than Π^0_1-CA_0^-, as $M := \{\langle x, e\rangle : \pi^0_1(x, e)\}$ is an ω-model of Π^0_1-CA_0^- with $M \dot\in M$ ($\pi^0_1(x, e)$ is a universal Π^0_1-formula with the property that for each Π^0_1-formula $A(u, v)$ with at most the displayed variables free, Π^0_1-$\mathsf{CA}_0^- \vdash \forall y\, \exists e \forall x[A(y, x) \leftrightarrow \pi^0_1(x, e)]$; cf. e.g. Simpson [26], Definition VII.1.3).*

Due to the above remark, we have to consider the following variant of the operation p_1.

Definition A.2.5. $\tilde{\mathsf{p}}_1(\breve{\mathsf{T}}) := \forall Z \exists X (Z \dot{\in} X \wedge \overline{Z} \dot{\in} X \wedge \breve{\mathsf{T}} \restriction X) \wedge \mathsf{pair} \wedge \mathsf{trans}$.

Note that ACA_0 proves $\tilde{\mathsf{p}}_1((\Pi^0_1\text{-}\mathsf{CA})^-)$, as for $X := \{\langle x, e\rangle : \pi^0_1(Z, x, e)\}$, $Z \dot{\in} X$ and $(\Pi^0_1\text{-}\mathsf{CA})^- \restriction X$. Further, by Lemma A.2.3, $\tilde{\mathsf{p}}_1(\Pi^0_1\text{-}\mathsf{CA}_0^-)$ proves arithmetical comprehension. Therefore, we could present ACA_0 as $\tilde{\mathsf{p}}_1(\Pi^0_1\text{-}\mathsf{CA}_0^-)$, and choose (ACA) to be the Π^1_2-sentence $\tilde{\mathsf{p}}_1((\Pi^0_1\text{-}\mathsf{CA})^-)$.

Bibliography

[1] Beckmann, A. *Separating fragments of bounded arithmetic.* PhD thesis, Universität Münster, 1996.

[2] Buchholz, W., Feferman, S., Pohlers, W., and Sieg, W. *Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies*, vol. 897 of *Lecture Notes in Mathematics.* Springer, Berlin, 1981.

[3] Feferman, S. Reflecting on incompleteness. *Journal of Symbolic Logic 56*, 1 (1991), 1–49.

[4] Gentzen, G. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, Neue Folge 4* (1938), 19–44.

[5] Gibbons, B. *The Veblen Hierarchy explained via Mahlo Hierachies in Constructive Set Theory.* PhD thesis, University of Leeds, 2003.

[6] Jäger, G. Metapredicative and explicit Mahlo: a proof-theoretic perspective. In *Proceedings of Logic Colloquium '00* (2005), R. Cori, A. Razborov, S. Todorcevic, and C. Wood, Eds., vol. 19 of *Association of Symbolic Logic Lecture Notes in Logic*, AK Peters, AK Peters, pp. 272–293.

[7] Jäger, G., Kahle, R., Setzer, A., and Strahm, T. The proof-theoretic analysis of transfinitely iterated fixed point theories. *Journal of Symbolic Logic 64*, 1 (1999), 53–67.

[8] Jäger, G., and Strahm, T. Fixed point theories and dependent choice. *Archive for Mathematical Logic 39* (2000), 493–508.

[9] Jäger, G., and Strahm, T. Upper bounds for metapredicative Mahlo in explicit mathematics and admissible set theory. *The Journal of Symbolic Logic 66*, 2 (2001), 935–958.

[10] Jäger, G., and Strahm, T. Reflections on reflections in explicit mathematics. *Annals of Pure and Applied Logic 136*, 1–2 (2005), 116–133. Festschrift on the occasion of Wolfram Pohlers' 60th birthday.

[11] Pohlers, W. *Proof Theory: An Introduction*, vol. 1407 of *Lecture Notes in Mathematics*. Springer, Berlin, 1989.

[12] Probst, D. *Pseudo-Hierarchies in Admissible Set Theories without Foundation and Explicit Mathematics*. PhD thesis, Universität Bern, 2005.

[13] Ranzi, F. *From a Flexible Type System to Metapredicative Wellordering Proofs*. PhD thesis, Institut für Informatik, Universität Bern, 2015.

[14] Ranzi, F., and Strahm, T. A flexible type system for the small Veblen ordinal. Submitted.

[15] Rathjen, M. The realm of ordinal analysis. In *Sets and Proofs*, S. B. Cooper and J. Truss, Eds. Cambridge University Press, 1999, pp. 219–279.

[16] Rathjen, M. The strength of Martin-Löf type theory with a superuniverse. Part I. *Archive for Mathematical Logic 39*, 1 (2000), 1–39.

[17] Rathjen, M. The strength of Martin-Löf type theory with a superuniverse. Part II. *Archive for Mathematical Logic 40*, 3 (2001), 207–233.

[18] Rathjen, M., and Vizcaíno, P. F. V. Well ordering principles and bar induction. In *Gentzen's Centenary: The Quest for Consistency*, R. Kahle and M. Rathjen, Eds. Springer, 2015, pp. 533–561.

[19] Rüede, C. The proof-theoretic analysis of Σ^1_1 transfinite dependent choice. *Annals of Pure and Applied Logic 121*, 1 (2003), 195–234.

[20] Rüede, C. Universes in metapredicative analysis. *Archive for Mathematical Logic 42* (2003), 129–151.

[21] Schütte, K. *Beweistheorie*. Springer, 1960.

[22] Schütte, K. *Proof Theory*. Springer, Berlin, 1977.

[23] Schwichtenberg, H. Proof theory: Some applications of cut-elimination. In *Handbook of Mathematical Logic*, J. Barwise, Ed. North Holland, Amsterdam, 1977, pp. 867–895.

[24] Setzer, A. An introduction to well-ordering proofs in Martin-Löf's type theory. In *Twenty-five years of constructive type theory*, G. Sambin and J. Smith, Eds., vol. 36 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1998, pp. 245–263.

[25] Setzer, A. Ordinal systems. In *Sets and Proofs* (Cambridge, 1999), C. B. and J. Truss, Eds., Cambridge University Press, pp. 301 – 331.

[26] Simpson, S. G. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, 1998.

[27] Strahm, T. First steps into metapredicativity in explicit mathematics. In *Sets and Proofs*, S. B. Cooper and J. Truss, Eds. Cambridge University Press, 1999, pp. 383–402.

[28] Strahm, T. Autonomous fixed point progressions and fixed point transfinite recursion. In *Logic Colloquium '98*, S. Buss, P. Hájek, and P. Pudlák, Eds., vol. 13 of *Lecture Notes in Logic*. Association for Symbolic Logic, 2000, pp. 449–464.

[29] Strahm, T. Wellordering proofs for metapredicative Mahlo. *The Journal of Symbolic Logic 67*, 1 (2002), 260–278.

[30] Tait, W. Normal derivability in classical logic. In *The Syntax and Semantics of Infinitary Languages*, J. Barwise, Ed. Springer, Berlin, 1968, pp. 204–236.

[31] Takeuti, G. *Proof Theory*. North-Holland, Amsterdam, 1975.

[32] Takeuti, G. *Proof Theory*, 2nd ed. North-Holland, Amsterdam, 1987.

[33] Thiel, K. *Metapredicative Set Theories and Provable Ordinals*. PhD thesis, University of Leeds, 2003.

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Notations and abbreviations